ELEMENTS OF CO-ORDINATE AND SOLID GEOMETRY

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U. N. DHUR & SONS PRIVATE LTD.
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ELEMENTS OF CO-ORDINATE
AND
SOLID GEOMETRY

[For Pre-University, Entrance and Higher Secondary Courses]

BY

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SECOND EDITION

U. N. DHUR & SONS, PRIVATE LTD.
15, BANKIM CHATTERJEE STREET,
CALCUTTA 12.

Published by
DWIJENDRANATH DHUR, B. L.
For U. N. DHUR & SONS, PRIVATE LTD.
15, Bankim Chatterjee Street, Calcutta 12.

Printed by
TRIDIBESH BASU
THE K. P. BASU PTG. WORKS,
11, Mohendra Gossain Lane, Calcutta 6

PREFACE

This book Elements of Co-ordinate and Solid Geometry has been written in accordance with the syllabus of the Pre-University Course of the Calcutta University, of Entrance Examination of the Burdwan University and of the Higher Secondary Course of the Board of Secondary Education, West Bengal.

Regarding the subject-matter we have tried to make the exposition clear and concise, without going into unnecessary details. A good number of typical examples have been worked out by way of illustrations and examples for exercise have been selected very carefully.

Important formulæ have been given at the beginning of the book for ready reference. A few questions of the recent years are given at the end to give the students an idea of the standard of examination.

It is hoped that the book will meet the requirements of those for whom it is intended and we shall deem our labour amply rewarded if the students find the book useful to them.

We are taking this opportunity of thanking Prof. Tapen Moulik M. Sc. of the B. E. College, Shibpur, for his suggestions and help in the speedy publication of the book. Our thanks are also due to the authorities and the staff of the K. P. Basu Printing Works, Calcutta, who kindly completed the printing of the book in short period.

Corrections of misprints and suggestions for improvement will be thankfully received.

15th April, 1962
Calcutta

B. C. D.
B. N. M.

Syllabus of Co-ordinate Geometry

1. Pre-University Course of the Calcutta University.

Plane cartesian co-ordinates, distance between two points, co-ordinates of the point dividing a finite straight line in a given ratio. Area of a triangle.

Equation of a locus in rectangular cartesian co-ordinates. Transfer of origin without rotation of axes. Equations of a straight line in different forms. Angle between two straight lines, conditions for parallelism and perpendicularity. Perpendicular distance of a point from a given line. Equations of the angle bisectors between two lines. Equation of a circle. Intersection of a straight line and a circle. Condition of tangency. Equation of the tangent and normal at a point.

Definition of a circle with reference to focus and directrix; a parabola, an ellipse and a hyperbola. Deduction, from definition, of the equations of the above loci referred to the directrix and the perpendicular from the focus upon the directrix as axes. Reduction of these equations to their standard forms. Intersection of a straight line with any of above loci; condition for tangency. Equation of tangent and normal at a point for each of the above loci. Deduction of simple properties of the above loci.

2. Higher Secondary Course of the Board of Secondary Education, West Bengal.

Class X

Rectangular cartesian co-ordinates in a plane; Lengths of segments; Section of a finite segment in a given ratio; Area of a triangle; Straight line.

Class XI

Circle, chords, tangents, Normals and elementary properties connected with them; Parabola, Ellipse, Hyperbola referred to their principal axes. Analytical treatment of those curves in respect of (1) the focus and directrix properties, (2) tangents and normals and elementary properties connected with them, (3) centre and diameter. [Note : Discussion should always be restricted to rectangular cartesian co ordinates.]

Syllabus of Solid Geometry

1. Pre-University Course of the Calcutta University.

Definitions—Parallel and skew straight lines. Angle between two straight lines and between two planes, parallelism and perpendicularity, Angle between a plane and a straight line, their parallelism and perpendicularity. Projection of a line on a plane.

Axioms : (i) One and only one plane passes through a given line and a given point outside it.

(ii) If two planes have one point in common, they have at least a second point in common.

Theorems : (i) Two intersecting planes cut one another in a straight line and in no point outside it.

(ii) If a straight line is perpendicular to each of two intersecting lines at their point of intersetion, it is perpendicular to the plane in which they lie.

(iii) All straight lines drawn perpendicular to a given straight line at a given point on it are co-planar.

(iv) If of two parallel straight lines one is perpendicular to a plane, the other is also perpendicular to it.

(v) If a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to that plane.

Idea of the following regular solids :

Sphere, rectangular parallelopiped, regular tetrahedron, right prism, right circular cylinder and a right cone. Expressions (without proof) for the surfaces and volumes of the above solids.

2. Higher Secondary Course of the Board of Secondary Education, West Bengal.

Axiom (i) : One and only one plane may be made to pass through any two intersecting lines.

Axiom (ii) : Two intersecting planes cut one another in a straight line and in no point outside it.

To prove.

1. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, it is also perpendicular to the plane in which they lie.

2. All straight lines drawn perpendicular to a given straight line at a given point on it are co-planar.

3. If two straight lines are parallel and if one of them is perpendicular to a plane, then the other is also perpendicular to the plane.

Concept of angle between two planes and angle between a straight line and a plane.

Concept of parallelism of planes.

Concept of a line being parallel to a plane.

Concept of skew lines.

Mensuration :

Parallelopipeds, Right Circular cones, Prisms and Pyramids (Expressions without proof, of the surfaces and volumes of the solids).

CONTENTS

Co-ordinate Geometry

Solid Geometry

Important Formulæ
of
CO-ORDINATE GEOMETRY

1. Distance $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Distance $OP = r = \sqrt{x^2 + y^2}$.

2. Point dividing the line joining two given points in a given ratio :

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}.$$

Middle point $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$.

3. Area of a triangle with given vertices

$$\tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}.$$

4. General equation of a straight line

$$ax + by + c = 0 \ (a \text{ and } b \text{ both } \neq 0).$$

Every first degree equation in x, y represents a straight line.

5. Transfer of the origin (directions of axes remaining unchanged) from (0, 0) to (α, β)

$$x = X + \alpha, \ y = Y + \beta.$$

6. Straight line parallel to the x-axis : $y = b$.

Straight line parallel to the y-axis : $x = a$.

7. Equations of straight lines in standard forms :

(i) Intercept form : $\frac{x}{a} + \frac{y}{b} = 1$.

(ii) 'm' form : $y = mx + c$.

(iii) Form through a given point :

$$y - y_1 = m(x - x_1), \ \text{or,} \ \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}.$$

(iv) Normal (or perpendicular) form :

$$x \cos \alpha + y \sin \alpha = p.$$

(v) Two points form : $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1} (x - x_1).$

8. Point of Intersection of the two lines

$$a_1x + b_1y + c_1 = 0, \; a_2x + b_2y + c_2 = 0 \;;$$

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}.$$

9. Condition for concurrence of the three given lines $a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0, \quad a_3x + b_3y + c_3 = 0$; $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0.$

10. Condition for collinearity of the three given points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, is

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

11. Angle between two given lines :

(i) When the lines are $y = m_1x + c_1$, $y = m_2x + c_2$

$$\tan \phi = \frac{m_1 \sim m_2}{1 + m_1m_2}.$$

(ii) When the lines are

$$a_1x + b_1y + c_1 = 0, \; a_2x + b_2y + c_2 = 0$$

$$\tan \phi = \frac{a_1b_2 \sim a_2b_1}{a_1a_2 + b_1b_2}.$$

12. Conditions for

(*a*) parallel lines, (i) $m_1 = m_2$, (ii) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$.

(*b*) perpendicular lines, (i) $m_1m_2 = -1$,

(ii) $a_1a_2 + b_1b_2 = 0.$

13. Length of the perpendicular from the point (x_1, y_1) upon the line $ax + by + c = 0$ is

$$\pm \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

14. Equations of the bisectors of the angle between the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}.$$

15. Equation of the circle

(i) Standard form : $x^2 + y^2 = a^2$

centre : $(0, 0)$; radius a.

(ii) general form : $x^2 + y^2 + 2gx + 2fy + c = 0$

centre : $(-g, -f)$, radius $= \sqrt{g^2 + f^2 - c}$.

16. Circle with the given points (x_1, y_1) and (x_2, y_2) as extremities of a diameter

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

17. Equation of the tangent to the circle at (x_1, y_1)

(i) for standard form : $xx_1 + yy_1 = a^2$,

(ii) for general form :

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

18. Equation of the normal to the circle at (x_1, y_1)

(i) for standard form : $\dfrac{x}{x_1} = \dfrac{y}{y_1}$.

(ii) for general form : $x(y_1 + f) - y(x_1 + g) = fx_1 - gy_1$.

19. Length of the chord of the circle $x^2 + y^2 = a^2$ intercepted by the line $y = mx + c$ is

$$2\,\frac{\sqrt{a^2(1 + m^2) - c^2}}{\sqrt{1 + m^2}}.$$

20. Condition of tangency : condition that the line $y = mx + c$ may touch the circle $x^2 + y^2 = a^2$ is

$$c = \pm a\sqrt{1 + m^2}.$$

$y = mx + a\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = a^2$ for all values of m and in that case the point of contact is

$$-\frac{am}{\sqrt{1+m^2}}, \quad \frac{a}{\sqrt{1+m^2}}.$$

21. Length of the tangent from an external point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\sqrt{x_1{}^2 + y_1{}^2 + 2gx_1 + 2fy_1 + c}.$$

22. Standard forms of the equations of conics.

(*a*) Parabola

(i) $y^2 = 4a(x-a)$ (with axis and directrix as axes of co-ordinates).

(ii) $y^2 = 4ax$ (Standard form), (with the vertex as origin and the axis and the tangent at the vertex as axes of co-ordinates).

(*b*) Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ (Standard form).

(with centre as origin, and major and minor axes as axes of co-ordinates).

(*c*) Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Standard form)

(with centre as origin and transverse and conjugate axes as axes of co-ordinates).

23. Parabola :

(i) Standard form $y^2 = 4ax$.

(ii) Latus rectum $= 4a$; focus is $(a, 0)$; extremities of the latus rectum are $(a, \pm 2a)$; directrix is $x = -a$.

(iii) Equation of the tangent at (x_1, y_1) is
$yy_1 = 2a(x + x_1)$.

(iv) Normal at (x_1, y_1) is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$.

(v) Length of the chord intercepted by the straight line $y = mx + c$ is $\frac{4}{m^2}\sqrt{a(a - mc)(1 + m^2)}$.

(vi) Condition that $y = mx + c$ may touch the parabola is $c = \frac{a}{m}$ $(m \neq 0)$.

The line $y = mx + \frac{a}{m}$ is a tangent to the parabola for all values of m (except zero),
the point of contact being $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

(vii) Parametric representation : $x = at^2$, $y = 2at$.

(viii) Equation of the diameter : $y = \frac{2a}{m}$.

24. Ellipse

(i) Standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(ii) Latus rectum $= 2a(1 - e^2) = 2\frac{b^2}{a}$.

(iii) Eccentricity : $b^2 = a^2(1 - e^2)$ or $e^2 = \frac{a^2 - b^2}{a^2}$.

(iv) Focal distances of $P(x_1, y_1)$:
$SP = a - ex_1$, $S'P = a + ex_1$; $SP + S'P = 2a$.

(v) Tangent at (x_1, y_1) : $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.

(vi) Normal at (x_1, y_1) : $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{\frac{y_1}{b^2}}$.

(vii) Length of the chord intercepted by the line $y=mx+c$ on the ellipse

$$=\frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2+b^2-c^2}}{a^2m^2+b^2}.$$

(viii) Condition of tangency :

The line $y=mx+c$ is a tangent to the ellipse if $c=\pm\sqrt{a^2m^2+b^2}$.

The line $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse for all values of m, and the point of contact is

$$-\frac{a^2m}{\sqrt{a^2m^2+b^2}},\ \frac{b^2}{\sqrt{a^2m^2+b^2}}.$$

(ix) Auxiliary circle : $x^2+y^2=a^2$.

(x) Parametric representation : $x=a\cos\theta$, $y=b\sin\theta$.

(xi) Diameter $y=-\dfrac{b^2}{a^2m}x$.

(xii) Director circle $x^2+y^2=a^2+b^2$.

25. Hyperbola

(i) Standard equation : $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

(ii) Latus rectum : $2a(e^2-1)=2\dfrac{b^2}{a}$.

(iii) Eccentricity : $b^2=a^2(e^2-1)$ or $e^2=\dfrac{a^2+b^2}{a^2}$.

For rectangular (or equilateral) hyperbola $a=b$; $e=\sqrt{2}$.

(iv) Focal distances of $P(x_1, y_1)$

$SP=ex_1-a$, $S'P=ex_1+a$

$S'P-SP=2a$.

(v) Equation of the tangent at (x_1, y_1)

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(vi) Equation of the normal at (x_1, y_1) is

$$\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}.$$

(vii) Length of the chord of the hyperbola intercepted by $y = mx + c$ is

$$\frac{2ab\sqrt{1+m^2}\sqrt{c^2 - a^2m^2 + b^2}}{a^2m^2 - b^2}.$$

(viii) Condition of tangency :

The line $y = mx + c$ will be a tangent to the hyperbola if $c = \pm\sqrt{a^2m^2 - b^2}$.

The line $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent to the hyperbola for all values of m, the point of contact being $\left(-\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, -\frac{b^2}{\sqrt{a^2m^2 - b^2}}\right)$.

(ix) Equation of the diameter is $y = \frac{b^2}{a^2m}x$.

(x) Equation of the asymptotes : $y = \pm\frac{b}{a}x$.

Important Formulæ
of
SOLID GEOMETRY

1. *Rectangular parallelopiped* (or cuboid).

If a, b, c be its length, breadth and height

(i) Area of the surface $= 2(bc + ca + ab)$.

(ii) Volume $= abc$.

(iii) Surface area of a cube of side $a = 6a^2$.

(iv) Volume $= a^3$.

2. *Right Pyramid on any regular base.*

(i) Slant surface $= \frac{1}{2}$ (perimeter of base) × slant height.

(ii) Volume $= \frac{1}{3}$ (area of base) × height.

3. *Tetrahedron.*

Volume $= \frac{1}{3}$ (area of base) × height.

4. *Right Prism.*

(i) Lateral surface = (perimeter of base) × height.

(ii) Volume = (area of base) × height.

5. *Right circular cylinder.*

If r is the radius of the base and h the height of the cylinder,

(i) Area of the curved surface

= (circumference of base) × height

$= 2\pi rh$.

(ii) Area of the *whole surface*

$= 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$.

(iii) Volume = (area of base) × height $= \pi r^2 h$.

6. *Right circular cone.*

If r is the radius of the base, h the height, l the slant side and α the semi-vertical angle of the cone,

(i) Area of curved surface

$$= \tfrac{1}{2}(\text{circumference of base}) \times \text{slant side}$$
$$= \tfrac{1}{2}.2\pi r.l = \pi rl$$
$$= \pi r\sqrt{h^2+r^2} = \pi r^2 \operatorname{cosec} \alpha.$$

(ii) Area of the *whole surface* $= \pi r(l+r)$.

(iii) Volume $= \frac{1}{3}$ (area of base) $\times$ height

$$= \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi h^3 \tan^2 \alpha.$$

7. *Sphere.*

If r be the radius of the sphere,

(i) Area of curved surface $= 4\pi r^2$.

(ii) Volume $= \frac{4}{3}\pi r^3$.

CO-ORDINATE GEOMETRY

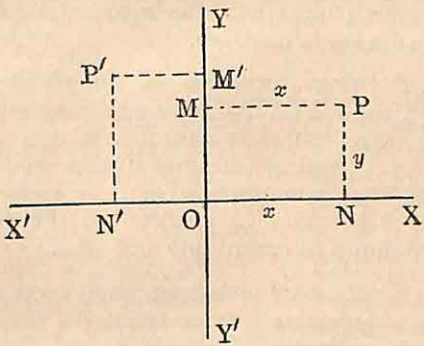
CHAPTER I

RECTANGULAR CARTESIAN CO-ORDINATES ; ELEMENTARY RESULTS

1·1. Location of a point on a plane.

Rectangular Cartesian Co-ordinates.

To locate the position of a point on a plane, we usually take two fixed straight lines on the plane, intersecting one another at right angles, for reference.



These lines are termed the *axes of reference* or *axes of co-ordinates*. One of them, XOX' (from right to left) is usually taken as the *x-axis*, and the other, YOY' (from above downwards) is taken as the *y-axis*. The point of intersection, O, is referred to as the *origin*. The position of a point P on the plane will be definitely known when we know its perpendicular distances PM and PN from the axes of reference. These perpendicular distances, with proper sign are termed the *co-ordinates* or more precisely the rectangular or orthogonal co-ordinates of the point.

The distance MP parallel to the x-axis, of P from the y-axis is called the x-co-ordinate, or *abscissa* of P, and is conventionally measured positive from O along OX towards the right, distances from O towards X' being reckoned as negative. Similarly, the distance NP parallel to the y-axis, from the x-axis to the point, is called the y-co-ordinate or the *ordinate* of P, being conventionally measured positive upwards along OY and negative downwards along OY'.

The whole plane is divided by the axes XOX' and YOY' into four quadrants XOY, YOX', $X'OY'$ and $Y'OX$, and according to the conventions given above, the co-ordinates of a point P are both positive in the first quadrant. In the second quadrant, (for instance for the point P'), the x-co-ordinate is negative and the y-co-ordinate is positive. Both the co-ordinates are negative in the third quadrant. while in the fourth quadrant, the x-co-ordinate is positive but the y-co-ordinate is negative.

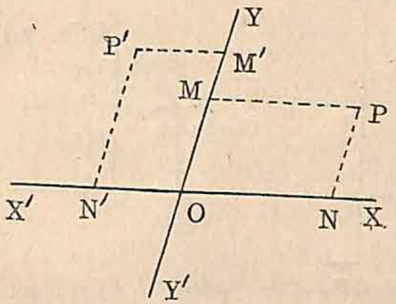
Generally, P being any point on the plane of axes of reference XOX' and YOY', PN being drawn perpendicular to the x-axis, ON is the x-co-ordinate and NP is the y-co-ordinate of P, and these co-ordinates being given in magnitude and sign, the position of P is definitely fixed. Conversely, if P be given in position on the plane, its co-ordinates are definite in magnitude and sign.

The above method of locating a point on a plane is due to Des Cartes, after whom the co-ordinates are referred to as Cartesian Co-ordinates (*Rectangular* or, *Orthogonal*) in this case.

Oblique co-ordinates.

Instead of two mutually perpendicular lines, we may take any two intersecting lines XOX' and YOY', inclined at any angle to one another as axes of reference and proceed to define the position of a point P, not by its perpendicular distances from the axes, but by the distances MP and NP of P parallel to $X'OX$ and $Y'OY$ respectively from the other axis. In other words, if PN be drawn parallel to YOY' to meet XOX' at N, ON and NP are respectively the x-co-ordinate

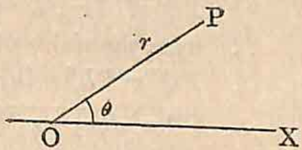
and the y-co-ordinate of P, which define the position of P on the plane definitely. The axes are here termed as *oblique axes* and the co-ordinates are *oblique co-ordinates* (Cartesian). The convention



as to the signs of the co-ordinates is exactly as in the case of rectangular axes.

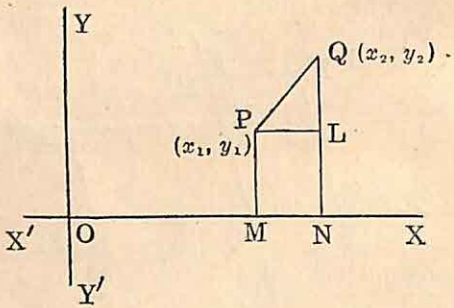
Polar Co-ordinates.

There is yet another method of locating a point on a plane. Here we take a fixed straight line OX on the plane for reference, which we call the *initial line, and* a fixed point O on it, called the *origin* or *pole*. The position of a point P is definitely known if we know the distance OP ($=r$ say), and the angle XOP ($=\theta$ say), The quantities r and θ are called the *polar co-ordinates* of P. The distance r, called the *radius vector*, is always taken as positive, and the angle θ, called the *vectorial angle* is conventionally positive when measured anti-clockwise from OX. The polar co-ordinates being given, the position of P is definite, while if the position of P be given, its polar co-ordinates will have definite values.



Henceforth, throughout the book, we shall use rectangular Cartesian co-ordinates only to refer to positions of points on a plane.

1·2. Distance between two points whose Cartesian co-ordinates are given.



Let P and Q be two given points on a plane whose Cartesian co-ordinates (rectangular) are (x_1, y_1) and (x_2, y_2) respectively. Let PM and QN be perpendiculars from P and Q on the x-axis OX, and PL perpendicular from P on NQ. Then $OM = x_1$, $MP = y_1$, $ON = x_2$, $NQ = y_2$.

$$\therefore \quad PL = MN = ON - OM = x_2 - x_1$$

$$LQ = NQ - NL = NQ - MP = y_2 - y_1$$

$\therefore$ from the right-angled triangle PLQ,

$$PQ^2 = PL^2 + LQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore \quad \mathbf{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

Cor. The distance r of a point P whose co-ordinates are given to be x, y, from the origin (whose co-ordinates are 0, 0) is given by

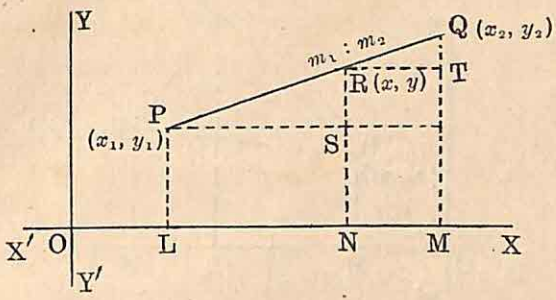
$$OP \equiv \mathrm{r} = \sqrt{\mathrm{x}^2 + \mathrm{y}^2}.$$

1·3. Point dividing the line joining two given points in a given ratio.

Let P and Q be two given points whose co-ordinates are (x_1, y_1) and (x_2, y_2) respectively and let R be the point which divides PQ in a given ratio $m_1 : m_2$, *i.e.*, $PR : RQ = m_1 : m_2$.

Let (x, y) be the co-ordinates of R.

Draw PL, QM and RN perpendiculars on the x-axis, and let PS and RT be parallel to the x-axis, meeting RN



and QM at S and T respectively. Then the triangles PSR and RTQ are evidently similar.

Hence, $\dfrac{PS}{RT}=\dfrac{SR}{TQ}=\dfrac{PR}{RQ}=\dfrac{m_1}{m_2}$.

But $PS=ON-OL=x-x_1$, $RT=OM-ON=x_2-x$.

Similarly, $SR=y-y_1$, $TQ=y_2-y$.

$$\therefore \quad \frac{x-x_1}{x_2-x}=\frac{y-y_1}{y_2-y}=\frac{m_1}{m_2}.$$

Hence, simplifying,

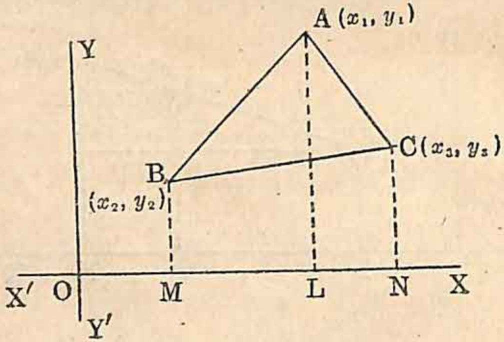
$$\mathbf{x}=\frac{\mathbf{m_1x_2+m_2x_1}}{\mathbf{m_1+m_2}},\ \mathbf{y}=\frac{\mathbf{m_1y_2+m_2y_1}}{\mathbf{m_1+m_2}},$$

giving the co-ordinates of R.

Cor. The co-ordinates of the **middle point** of the line joining the points (x_1, y_1) and (x_2, y_2) is $\frac{1}{2}(\mathbf{x_1+x_2})$, $\frac{1}{2}(\mathbf{y_1+y_2})$.

Note. If R divides PQ *externally* in the ratio $m_1 : m_2$, one of the two quantities m_1 and m_2 is to be taken with a *negative* sign. Whichever of the two is taken negative, the co-ordinates of R will remain the same namely $\dfrac{m_1x_2-m_2x_1}{m_1-m_2}$, $\dfrac{m_1y_2-m_2y_1}{m_1-m_2}$.

1·4. Area of a triangle whose vertices are given.



Let the vertices A, B, C of a given triangle have co-ordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively.

Let AL, BM, CN be drawn perpendiculars on the x-axis, so that $OL=x_1$, $LA=y_1$, $OM=x_2$, $MB=y_2$, $ON=x_3$, $NC=y_3$.

Now evidently,

$$\triangle ABC = \text{trapezium } ABML + \text{trapezium } ALNC - \text{trapezium } BMNC.$$

Also, area of a trapezium $=\frac{1}{2}$ the sum of the parallel sides $\times$ the perpendicular distance between them.

Hence, the area of the triangle ABC is given by

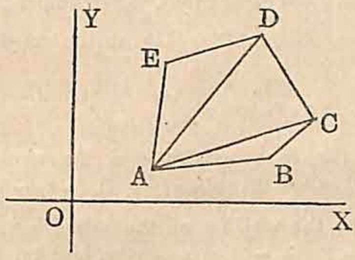
$$\begin{aligned}\triangle &= \tfrac{1}{2}(MB+LA)\times ML+\tfrac{1}{2}(LA+NC)\times LN - \tfrac{1}{2}(MB+NC)\times MN\\ &= \tfrac{1}{2}(y_2+y_1)(x_1-x_2)+\tfrac{1}{2}(y_1+y_3)(x_3-x_1) - \tfrac{1}{2}(y_2+y_3)(x_3-x_2)\\ &= \tfrac{1}{2}\{\mathbf{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)}\}\end{aligned}$$

on simplification.

Note 1. In our figure above, we have taken the vertices A, B, C in such an order that in moving along the sides AB, BC, CA we are to move in an anti-clockwise direction. In such a case, the above

expression for the area of the triangle ABC will always be found to be *positive*. If the vertices in order are so taken that in moving along the sides of the triangle in that order we move clockwise, the above expression for the area will be found to be *negative*.

Note 2. Area of a polygon with given vertices.



If a polygon $ABCDE$ has the co-ordinates of its vertices given, to find its area, we break it up into triangles by joining its diagonals, and then add up the areas of these triangles (which have got their vertices known) in the same order. The result will give the required area of the polygon in that order.

1·5. Illustrative Examples.

Ex. 1. *Prove that* $(-1, -1)$, $(1, 1)$ *and* $(-\sqrt{3}, \sqrt{3})$ *are the vertices of an equilateral triangle.*

Let A, B, C be the points whose co-ordinates are $(-1, -1)$ $(1, 1)$ and $(-\sqrt{3}, \sqrt{3})$ respectively.

Then $\quad AB^2=(-1-1)^2+(-1-1)^2=8$

$$BC^2=(1+\sqrt{3})^2+(1-\sqrt{3})^2=8$$

$$CA^2=(-\sqrt{3}+1)^2+(\sqrt{3}+1)^2=8$$

$\therefore\ AB^2=BC^2=CA^2$ or $AB=BC=CA$.

Thus the triangle ABC is equilateral.

Ex. 2. *If the co-ordinates of* A, B, C, D *are* $(-1, -2)$, $(7, 4)$, $(4, 8)$ *and* $(-4, 2)$ *respectively, show that* $ABCD$ *is a rectangle.*

Here, $AB^2=(-1-7)^2+(-2-4)^2=100$
$BC^2=(7-4)^2+(4-8)^2=25$
$CD^2=(4+4)^2+(8-2)^2=100$
$DA^2=(-4+1)^2+(2+2)^2=25$
and $AC^2=(-1-4)^2+(-2-8)^2=125.$

Thus, $AB=CD$ and $BC=DA$.

Hence, opposite sides being equal, $ABCD$ must be a parallelogram.

Moreover, here, $AC^2=AB^2+BC^2$, so that angle ABC is a right angle.

Hence, $ABCD$ is a rectangle.

Ex. 3. *Find the circum-centre and the circum-radius of the triangle whose vertices are the points* $(-4, 3)$, $(-2, -3)$, $(0, -5)$.

Let A, B, C be the vertices of the triangle whose co-ordinates are $(-4, 3)$, $(-2, -3)$, $(0, -5)$ respectively, and let S be the circum-centre of the triangle, whose co-ordinates are supposed to be (x, y).

Then, $SA=SB=SC$ or $SA^2=SB^2=SC^2$.

$\therefore \quad (x+4)^2+(y-3)^2=(x+2)^2+(y+3)^2=(x-0)^2+(y+5)^2,$

or $\quad 8x-6y+25=4x+6y+13=10y+25,$

whence solving, $x=6$, $y=3$.

$\therefore$ the circum-centre S is the point $(6, 3)$.

Also, circum-radius $=SA=\sqrt{(6+4)^2+(3-3)^2}=10$ units.

Ex. 4. *A, B, C, D are four points whose co-ordinates are* $(1, -8)$, $(-3, 4)$, $(0, 7)$ *and* $(3, 16)$ *respectively. Find the ratio in which CD divides AB. Find also the ratio in which AB divides CD.*

Let AB and CD intersect each other at P which divides AB in the ratio $\lambda : 1$.

Then, the co-ordinates of P (by § 1·3) are

$$\frac{-3\lambda+1}{\lambda+1}, \frac{4\lambda-8}{\lambda+1} \text{ respectively.}$$

Now, P, C, D being in one straight line, the area of the triangle PCD is zero. Hence (by § 1·4).

$$\frac{1}{2}\left[\frac{-3\lambda+1}{\lambda+1}(7-16)+0\cdot\left(16-\frac{4\lambda-8}{\lambda+1}\right)+3\left(\frac{4\lambda-8}{\lambda+1}-7\right)\right]=0,$$

or, $\quad -9(-3\lambda+1)+3(-3\lambda-15)=0\,; \quad \therefore\ \lambda=3.$

Hence, CD divides AB in the ratio 3 : 1.

The co-ordinates of P are therefore,

$$\frac{3(-3)+1}{3+1},\ \frac{3.4-8}{3+1}\ i.e.,\ -2,\ 1 \text{ respectively.}$$

Again, if P divides CD in the ratio $\mu:1$, then co-ordinates of P are $\frac{\mu.3+0}{\mu+1}$, $\frac{\mu.16+7.1}{\mu+1}$ and these being identical with $-2, 1$, we have

$$\frac{3\mu}{\mu+1}=-2 \text{ and } \frac{16\mu+7}{\mu+1}=1.$$

Either of these give $\mu=-\frac{2}{5}$.

Hence, AB divides CD externally in the ratio 2 : 5.

Ex. 5. *The vertices A, B, C of a triangle have co-ordinates* (5, 6), (−9, 1) and (−3, −1) *respectively. Find the length of the perpendicular from A on BC.*

The area of the triangle ABC is

$\frac{1}{2}[5(1+1)-9(-1-6)-3(6-1)]=29$ square units.

Also, distance $BC=\sqrt{(-9+3)^2+(1+1)^2}=\sqrt{40}$ units.

Hence, if p be the length of the perpendicular from A on BC, area of the triangle $ABC=\frac{1}{2}.p.\sqrt{40}=29.$

$$\therefore\ p=\frac{29\times 2}{\sqrt{40}}=\frac{29}{\sqrt{10}}=\frac{29}{10}\sqrt{10} \text{ units.}$$

Examples I

1. Find the distance between the following pair of points :

(i) (2, 1) and (8, 9). (ii) (−3, 5) and (9, −2).

(iii) (−6, −7) and (4, −9).

(iv) $(a\cos\alpha,\ a\sin\alpha)$ and $(a\cos\beta,\ a\sin\beta)$.

(v) $(at_1^2,\ 2at_1)$ and $(at_2^2,\ 2at_2)$.

2. Prove that the point (−2, −11) is equidistant from (−3, 7) and (4, 6).

3. Show that the triangle whose vertices are $(-2, -5)$, $(4, -1)$, $(-1, 0)$ is isosceles.

4. If the ordinate of a point equidistant from $(-3, 7)$ and $(6, -11)$ be 9, find its abscissa.

5. Prove that the points $(4, 8)$, $(4, 12)$ and $(4+2\sqrt{3}, 10)$ are the vertices of an equilateral triangle.

6. Show that the lines joining the point $(4, 5)$ to the points $(-2, -3)$ and $(16, -4)$ are at right angles.

7. Prove that the points $(-2, 3)$, $(-3, 10)$ and $(4, 11)$ are the angular points of an isosceles right-angled triangle.

8. Show that the lines joining successively the points $(-2, -5)$, $(7, -1)$, $(8, 6)$ and $(-1, 2)$ form a parallelogram.

9. Prove that the points $(2, -2)$, $(8, 4)$, $(5, 7)$ and $(-1, 1)$ are the successive angular points of a rectangle.

10. Show that (i) the points $(3, -5)$, $(9, 10)$, $(3, 25)$ and $(-3, 10)$ are the vertices of a rhombus ;

(ii) the points $(-a, -a)$, $(a, 0)$, $(0, 2a)$, $(-2a, a)$ are the vertices of a square.

11. (i) Prove that the point $(11, 2)$ is the circum-centre of the triangle whose vertices are $(1, 2)$, $(3, -4)$ and $(5, -6)$, and find the circum-radius of the triangle.

(ii) Find the circum-centre of the triangle whose vertices are $(1, 1)$, $(2, 3)$ and $(-2, 2)$.

12. A and B are two fixed points whose co-ordinates are $(2, 4)$ and $(2, 6)$ respectively ; ABP is an equilateral triangle on the side of AB opposite to the origin. Find the co-ordinates of P. [*H. S. 1961*]

13. In a triangle ABC, AD is the median bisecting BC. Prove analytically that

$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$

[*Choose rectangular axes with A as origin.*]

14. G is the centroid of a triangle ABC, and P is any other point on the plane. Prove that

(i) $BC^2 + CA^2 + AB^2 = 3(GA^2 + GB^2 + GC^2)$.

(ii) $PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3GP^2$.

[*Choose rectangular axes with G as origin.*]

15. Find the co-ordinates of the points which divide the line joining the points $(1, -2)$ and $(4, 1)$ in the ratio $2 : 1$ (i) internally, (ii) externally.

16. The co-ordinates of the points A, B, C, D are $(-2, 3)$, $(1, -2)$, $(8, -3)$ and $(5, 2)$ respectively. Show that AC and BD bisect each other.

17. Find the areas of the triangles whose vertices are respectively,

(i) $(3, -4)$, $(7, 5)$, $(-1, 10)$.

(ii) $(0, 4)$, $(3, 6)$, $(-8, -2)$.

(iii) $(5a, 0)$, $(-2a, 7a)$, $(-6a, -3a)$.

(iv) $(a, b+c)$, $(a, b-c)$, $(-a, c)$.

(v) $(a \cos \alpha, b \sin \alpha)$, $(a \cos \beta, b \sin \beta)$, $(a \cos \gamma, b \sin \gamma)$.

18. Prove that the area of the triangle whose vertices are the points $(at_1{}^2, 2at_1)$, $(at_2{}^2, 2at_2)$ and $(at_3{}^2, 2at_3)$ respectively is $a^2(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)$.

19. Show that the points $(1, 4)$, $(3, -2)$ and $(-3, 16)$ are collinear.

[*The triangle formed by them is of zero area.*]

20. Shew that the points $(0, -3)$, $(3, 0)$ and $(5, 2)$ lie on a straight line.

21. (i) Find the area of the triangle whose vertices A, B, C are respectively $(3, 4)$, $(-4, 3)$ and $(8, 6)$.

Hence, or otherwise find the length of the perpendicular from A on BC. [*H. S. 1961*]

(ii) The vertices A, B, C of a triangle ABC have co-ordinates $(5, 2)$, $(-9, 3)$ and $(-3, -5)$ respectively. Find the length of the perpendicular from A on BC.

22. Find the area of the quadrilateral whose angular points taken in order are $(2, 8)$, $(0, -7)$, $(8, -6)$ and $(6, 11)$.

23. Find the area of the quadrilateral whose angular points are respectively $(1, 1)$, $(3, 4)$, $(5, -2)$, $(4, -7)$.

[*C. U. 1944*]

24. (i) Show that the straight line joining the points $(0, -1)$ and $(15, 2)$ divides the line joining the points $(-1, 2)$ and $(4, -5)$ internally in the ratio $2 : 3$.

(ii) If A, B, C, D are points whose co-ordinates are $(-2, 3)$, $(8, 9)$, $(0, 4)$ and $(3, 0)$ respectively, and AB and CD are joined, find the ratio of the segments into which AB is divided by CD. [*H. S. 1960*]

ANSWERS

1. (i) 10. (ii) $\sqrt{193}$. (iii) $2\sqrt{26}$. (iv) $2a \sin \frac{1}{2}(\alpha \sim \beta)$.
(v) $a(t_1 \sim t_2)\sqrt{(t_1+t_2)^2+4}$.

4. 23·5. **11.** (i) 10. (ii) $-\frac{1}{14}, \frac{39}{14}$.

12. $2+\sqrt{3}$, 5. **15.** (i) 3, 0. (ii) 7, 4.

17. (i) 46 sq. units. (ii) 1 sq. unit. (iii) $49a^2$. (iv) $2ac$.
(v) $2ab \sin \frac{1}{2}(\beta-\gamma) \sin \frac{1}{2}(\gamma-\alpha) \sin \frac{1}{2}(\alpha-\beta)$.

21. (i) $4\frac{1}{2}$ sq. units ; $\frac{3}{17}\sqrt{17}$. (ii) 10·6.

22. 96 sq. units. **23.** $20\frac{1}{2}$ sq. units.

24. (ii) 11 : 47.

CHAPTER II

EQUATION AND LOCUS

2·1. Equation and Locus.

If an equation in x and y be given, the co-ordinates of any and every point on a plane will not generally satisfy the equation. On the other hand, there are innumerable points on the plane whose co-ordinates will satisfy the given equation ; for we may choose any value of x, and substituting that value in the given equation, solve for value or values of y whereby the given equation is satisfied. We thus get definite point or points with given x, whose co-ordinates satisfy the equation. As x may be chosen arbitrarily, the number of points will be infinite. These points joined will give us a line (straight or curved) on the plane, the co-ordinates of every point of which, (and of no point outside it), will satisfy the given equation. This line is defined as the *locus* represented by the given equation.

Conversely, if a line, straight or curved, be given on a plane, then if with chosen axes on the plane, a relation between x and y can be found which is satisfied by the co-ordinates of every point of the given line, and which holds for no other points except those lying on the line, then that relation between x and y is defined as the *equation* of the given line on the plane, with the chosen axes of reference.

For example, given the equation

$$x^2 + y^2 + 6x - 4y - 12 = 0,$$

we can write it in the form

$$(x+3)^2 + (y-2)^2 = 25,$$

or, $$\sqrt{\{x-(-3)\}^2 + (y-2)^2} = 5.$$

This relation shows that the distance of the point (x, y) from the fixed point $(-3, 2)$ is 5. [See § *1·2*]. Thus the different positions of the point (x, y) must be at a fixed distance 5 from the fixed point $(-3, 2)$. This identifies the

locus of the point (x, y) to be a circle with centre $(-3, 2)$ and radius 5. This is therefore the locus represented by the equation.

Conversely, suppose a straight line on a plane with chosen axes be given, passing through the given points $(3, 2)$ and $(-1, 5)$. Then if (x, y) be the co-ordinates of any arbitrary point on the line, the area of the triangle with (x, y), $(3, 2)$ and $(-1, 5)$ as vertices must be zero.

Hence, $\frac{1}{2}\{x(2-5)+3(5-y)+(-1)(y-2)=0.$

[See § *1·4*]

i.e., $-3x-4y+17=0$, or, $3x+4y=17$.

This being the relation satisfied by the co-ordinates of any and every point on the given line, it represents the equation to the given line.

2·2. Every first degree equation in x, y must represent a straight line.

Let $ax+by+c=0 \cdots$ (i) be a given equation of the first degree. Clearly, any equation of the first degree in x, y is of this type.

Take any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the locus. Any other point $P(x_3, y_3)$ on the locus being taken, the co-ordinates of A, B and P satisfy the given equation.

Hence, $ax_1+by_1+c=0 \cdots$ (ii)

$ax_2+by_2+c=0 \cdots$ (iii)

and $ax_3+by_3+c=0 \cdots$ (iv)

From (ii) and (iii), by cross multiplication,

$$\frac{a}{y_1-y_2}=\frac{b}{x_2-x_1}=\frac{c}{x_1y_2-x_2y_1}.$$

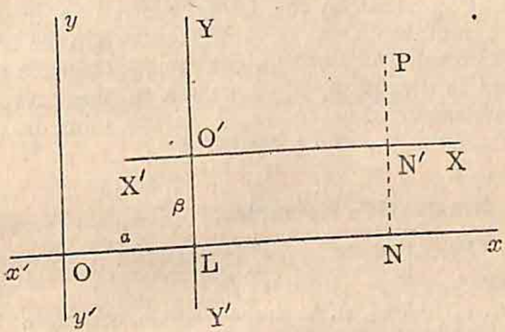
$\therefore$ substituting in (iv),

$$x_3(y_1-y_2)+y_3(x_2-x_1)+(x_1y_2-x_2y_1)=0$$

or, $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$

This shows that the triangle ABP is of zero area [See § 1·4]. Thus, P lies on the straight line AB. As P is any point on the locus represented by (i), it follows that the locus is a straight line.

2·3. Transfer of origin, directions of axes remaining unchanged.



Let O be the origin, and let, xOx', yOy' be a set of rectangular axes, with respect to which co-ordinates of a point P are x, y.

Let the origin be transferred to the point O' whose co-ordinates referred to the given axes xOx' and yOy' are (α, β), and let the axes $XO'X'$, and $YO'Y'$, which are parallel respectively to xOx' and yOy' through O' be taken as the new axes of co-ordinates. Suppose the co-ordinates of P referred to this new set of axes be X, Y.

Let PN be perpendicular to Ox and let it intersect $XO'X'$ at N'. Also let $YO'Y'$ intersect Ox at L.

Then evidently $ON = x$, $NP = y$, $O'N' = X$, $N'P' = Y$, $OL = \alpha$, $LO' = \beta$, in magnitude and sign.

Now, from the figure,

$$ON = OL + LN = OL + O'N'$$

and $NP = NN' + N'P = LO' + N'P$.

Hence, $\mathbf{x} = \mathbf{X} + \alpha$

$$\mathbf{y} = \mathbf{Y} + \beta.$$

These are the equations of transformation, connecting the old co-ordinates of P with the new ones.

These equations are easily seen to be true, in whichever quadrant O' may lie, provided in writing the relations from the figure, we take due notice of the signs of the quantities involved.

Thus, if an equation to a locus or curve be given referred to a set of rectangular axes, and if the origin be transferred to a position whose co-ordinates are α, β, the axes remaining unchanged in direction, the equation of the curve referred to the new axes will be obtained by replacing in the given equation x by $X+\alpha$ and y by $Y+\beta$.

2·4. Illustrative Examples.

Ex. 1. *Prove that the locus of a point which is always equidistant from two given points is a straight line.*

Let x_1, y_1 and x_2, y_2 be the co-ordinates of the two given points A and B, and let x, y be the co-ordinates of any point P which is equidistant from them. Then, $PA=PB$ or $PA^2=PB^2$.

$\therefore \quad (x-x_1)^2+(y-y_1)^2=(x-x_2)^2+(y-y_2)^2,$

or, $\quad 2(x_2-x_1).x+2(y_2-y_1).y+(x_1{}^2+y_1{}^2-x_2{}^2-y_2{}^2)=0.$

This being the equation satisfied by the co-ordinates x, y of any position of P, it represents the equation to the locus of P.

As this is a first degree equation in x, y, it represents a straight line [See § *2·2*]

Ex. 2. *A and B are two given points whose co-ordinates are* $(-5, 3)$ *and* $(2, 4)$ *respectively. A point P moves in such a manner that PA : PB* $=3:2$. *Find the equation to the locus traced out by P. What curve does it represent ?*

Let x, y be the co-ordinates for any position of P.

Then, $PA=\sqrt{(x+5)^2+(y-3)^2}$ and $PB=\sqrt{(x-2)^2+(y-4)^2}$. By the given condition, $PA:PB=3:2$ or $4PA^2=9PB^2$.

$\therefore \quad 4\{(x+5)^2+(y-3)^2\}=9\{(x-2)^2+(y-4)^2\},$

whence, $5(x^2+y^2)-76x-48y+44=0.$

As this is the equation satisfied by the co-ordinates of any position of P, it represents the required equation to the locus of P.

The equation may be put in the form

$$x^2+y^2-\tfrac{76}{5}x-\tfrac{48}{5}y+\tfrac{44}{5}=0,$$

or $$(x-\tfrac{38}{5})^2+(y-\tfrac{24}{5})^2=(\tfrac{38}{5})^2+(\tfrac{24}{5})^2-\tfrac{44}{5}=72,$$

or $$\sqrt{(x-7\tfrac{3}{5})^2+(y-4\tfrac{4}{5})^2}=6\sqrt{2}.$$

This shows that the distance of the moving point x, y from the fixed point $(7\frac{3}{5}, 4\frac{4}{5})$ is constant $=6\sqrt{2}$. This identifies the locus to be a circle with centre $(7\frac{3}{5}, 4\frac{4}{5})$ and radius $6\sqrt{2}$.

Ex. 3. *Prove that the equation*

$$9x^2+25y^2-108x+100y+199=0$$

by properly transferring the origin without turning the axes can be reduced to the form

$$\frac{x^2}{25}+\frac{y^2}{9}=1.$$

The given equation can be written as

$$9(x^2-12x)+25(y^2+4y)=-199,$$

or, $$9(x-6)^2+25(y+2)^2=9.36+25.4-199=225.$$

Now transferring the origin to $(6, -2)$ without turning the axes *i.e.*, replacing x by $x+6$ and y by $y-2$, [See § *2·3*] the new equation becomes

$$9x^2+25y^2=225, \text{ or, } \frac{x^2}{25}+\frac{y^2}{9}=1.$$

Examples II

1. A point moves so that (i) its distance from the y-axis is always -3, (ii) the sum of its distances from the axes of co-ordinates is always 9. Write down the equation to its locus in each case.

2. Find the equation to the locus of a point which moves so that

(i) its distance from the point -3, 7 is always equal to 4.

(ii) its distance from the y-axis is always half its distance from the orign.

(iii) the rectangle contained by its distances from the axes is equal to a square of side c.

3. If the difference of the squares of the distances of a moving point from two fixed points is constant, show that it traces a straight line.

4. A and B are two fixed points whose co-ordinates are $(-3, 4)$ and $(5, -2)$ respectively. A point P moves in such a manner that the area of the triangle PAB remains constant. Show analytically that its locus is a straight line.

5. If A, B, C are three fixed points whose co-ordinates are $(a, 0), (-a, 0), (c, 0)$ respectively, and if P be a movable point such that $PA^2 + PB^2 = 2PC^2$, find the equation to the locus of P.

6. The co-ordinates of three fixed points A, B, C are $(3, 4)$, $(-2, 5)$ and $(-1, -9)$ respectively. P is a variable point such that $PA^2 + PB^2 + PC^2 =$ constant. Prove that P traces out a circle.

7. A and B are two fixed points and P is a movable point such that $PA : PB$ is constant. Show that the locus of P is a circle.

[*Choose the middle point of AB as orign, and AB as x-axis.*]

8. A point moves so that its distance from the fixed point $(2a, 0)$ is equal to its distance from the y-axis. Determine the equation to its locus.

If the origin be transferred to $(a, 0)$, the axes remaining fixed in direction, prove that the transformed equation is $y^2 = 4ax$.

9. The sum of the distances of a movable point P from the two fixed points $(c, 0)$ and $(-c, 0)$ is a constant $= 2a$. Prove that the equation to the locus of P is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } a^2 - b^2 = c^2.$$

10. The distance between two fixed points S and S' is $2c$. Taking the middle point of SS' as origin, and SS' as x-axis, determine the equation to the locus of a point P which moves so that

$$PS - PS' = 2a \text{ (a constant)}.$$

11. Transform to parallel axes through the point $(1, -2)$ the equations

(i) $y^2 - 4x + 4y + 8 = 0.$

(ii) $2x^2 + y^2 - 4x + 4y = 0.$

12. What does the equation

$$(a-b)(x^2+y^2) - 2abx = 0$$

become if the origin be transferred to $\frac{ab}{a-b}$, 0, axes remaining unchanged in direction ?

13. The equation to a curve being

$$(1-e^2)\,x^2 + y^2 + d^2 = 2dx,$$

find its equation if the origin be shifted to $\frac{d}{1-e^2}$, 0, without rotation of axes.

14. By transforming to parallel axes through a properly chosen point, prove that the equation

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$$

can be reduced to a homogeneous equation of the second degree in x, y.

ANSWERS

1. (i) $x+3=0.$ (ii) $x+y=9.$ **2.** (i) $x^2+y^2+6x-14y+42=0.$

(ii) $y=\pm x\sqrt{3}.$ (iii) $xy=c^2.$ **5.** $x=\frac{c^2-a^2}{2c}.$

8. $y^2=4a(x-a).$ **10.** $\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1.$ **11.** (i) $y^2=4x.$

(ii) $\frac{x^2}{3}+\frac{y^2}{6}=1.$ **12.** $x^2+y^2=\left(\frac{ab}{a-b}\right)^2.$ **13.** $x^2+\frac{y^2}{1-e^2}=\left(\frac{de}{1-e^2}\right)^2.$

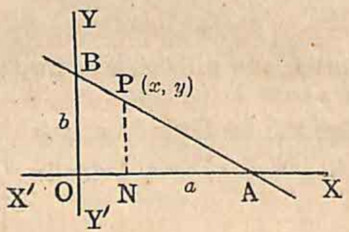
CHAPTER III

STRAIGHT LINES

3·1. Standard forms of the equation of a straight line.

In the previous chapter we have seen that a first degree equation always represents a straight line. Now the position of a given straight line on a plane may be fixed up in various ways, and according to each way of defining the position of a line, we get a definite form of the equation to the line. The standard forms of the equation to a line are given below :

(A) *Intercepts on the axes are given in magnitude and sign.*



Let a straight line AB intersect the axes of x and y at A and B respectively, and let the *intercepts* OA and OB on the axes be a and b (given in magnitude and sign) so that the position of the line is definite.

Let (x, y) denote the co-ordinates of any point P on the line. Let PN be the perpendicular from P on OX, so that $ON=x$, $NP=y$.

Now the triangles PNA and BOA are easily seen to be similar.

$$\therefore \quad \frac{NP}{OB}=\frac{NA}{OA}, \text{ or } \frac{y}{b}=\frac{a-x}{a}=1-\frac{x}{a}.$$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1$$

which being the relation satisfied by the co-ordinates x, y of any point on the line, is the equation to the line.

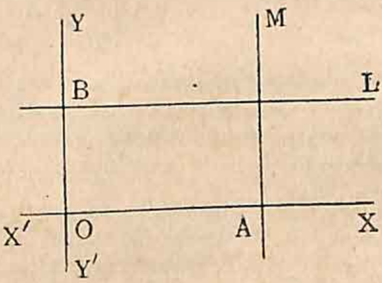
Alternatively

The co-ordinates of A and B are evidently (a, o) and (o, b) respectively. Now P being on the straight line AB, the triangle PAB is of zero area.

Hence, $\frac{1}{2}\{x(0-b)+a(b-y)+0(y-0)\}=0$,

or, $-xb+ab-ay=0$, whence $\dfrac{x}{a}+\dfrac{y}{b}=1$.

(B) *Straight lines parallel to x-axis or y-axis.*



For a straight line BL parallel to the x-axis, if its intercept $OB=b$, then clearly the distance of every point of it from the x-axis being the same, namely b, the y-co-ordinate of every point of it is b, whatever its x-co-ordinate may be. Hence, its equation is

$$y=b.$$

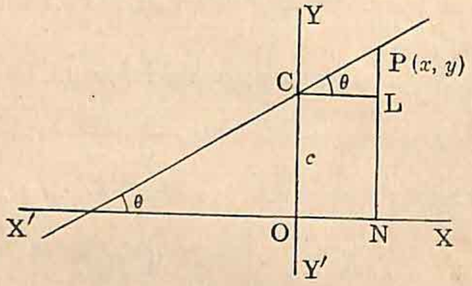
Similarly, for a straight line AM parallel to the y-axis, if its intercept OA on the x-axis be a, the equation will be

$$x=a.$$

Conversely, equations of the form $x=a$, or $y=b$, in which one co-ordinate is absent, represent straight lines parallel to the y-axis or to the x-axis, as the case may be.

Note. It may be noted that when a straight line becomes parallel to the y-axis, its intercept on the y-axis, $b \to \infty$, and so the equation to the line, from (A), reduces to the form $\frac{x}{a}=1$, *i.e.*, $x=a$. Similarly for lines parallel to the x-axis, the equation reduces to the form $y=b$.

(C) *Intercept on the y-axis, and inclination to the x-axis are given.*



Let a straight line, inclined at a given angle θ to the x-axis, intersect the y-axis at C, and let the intercept OC be given to be c (in magnitude and sign). θ and c being given, the position of the line is definitely fixed.

Let (x, y) be the co-ordinates of any point P on the line. PN being perpendicular from P on OX, we have $ON=x$, $NP=y$. Now CL being perpendicular from C on NP, $LP=NP-NL=NP-OC=y-c$, and $CL=ON=x$. Also $\angle PCL=\theta$.

$$\therefore \quad \tan\theta=\frac{LP}{CL}=\frac{y-c}{x}. \quad \therefore \quad y=x\tan\theta+c.$$

Denoting $\tan\theta$ by m,

$$\mathbf{y=mx+c \quad (m=\tan\theta).}$$

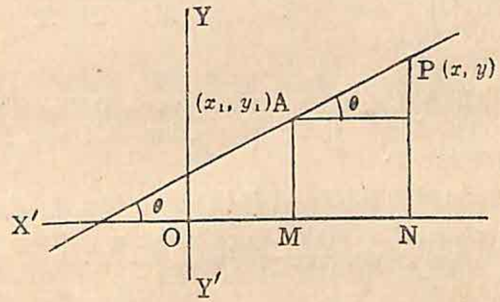
This being the relation satisfied by the co-ordinates of any point on the line, it represents the equation to the line.

Note. $m(=\tan\theta)$ is defined as the *slope* or *gradient* of the line. The angle θ may be acute or obtuse, and hence m may be positive or negative according to the position of the line. When the line is

parallel to the x-axis, $\theta=0$, and so $m=0$. The equation to the line then reduces to the form $y=c$ [See *(B) above*]. If $\theta=90°$, $m\to\infty$ or $\frac{1}{m}\to 0$. The equation to the line reduces to the form $x=a$ in this case [See *(B) above*].

(D) *Straight line through a given point* (x_1, y_1), *and making a given angle* θ *with the x-axis.*

Let A be the given point (x_1, y_1) and P any point (x, y) on the line, whose inclination to the x-axis is θ.



Let AM, PN be perpendiculars on OX, and AL perperpendicular on PN. Then, $\angle PAL=\theta$.

$$\text{Now, } \tan\theta=\frac{LP}{AL}=\frac{LP}{MN}=\frac{NP-MA}{ON-OM}=\frac{y-y_1}{x-x_1}.$$

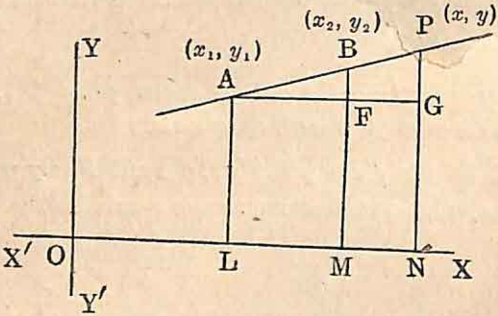
$$\therefore\ \mathbf{y-y_1}=(x-x_1)\tan\theta=\mathbf{m(x-x_1)}, \text{ where } m=\tan\theta,$$

$$\text{or, } \frac{\mathbf{x-x_1}}{\mathbf{\cos\theta}}=\frac{\mathbf{y-y_1}}{\mathbf{\sin\theta}},$$

any one of which represents the required equation to the line.

(E) *Straight line passing through two given points* (x_1, y_1) *and* (x_2, y_2).

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points through which a straight line passes, and $P(x, y)$ be any point on it.



Draw AL, BM, PN perpendiculars on OX, and let AFG be drawn parallel to OX, intersecting BM and PN at F and G respectively.

Then, triangles PAG and BAF are evidently similar.

$$\therefore \quad \frac{AG}{AF}=\frac{GP}{FB}, \text{ or, } \frac{ON-OL}{OM-OL}=\frac{NP-LA}{MB-LA}.$$

$$\therefore \quad \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1},$$

$$\text{or,} \quad y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1),$$

any one of which represents the required equation to the line.

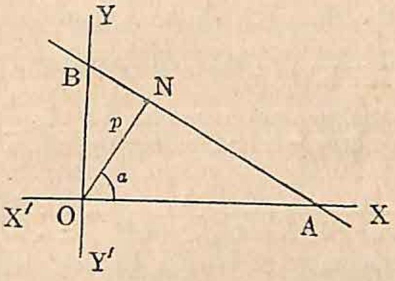
The slope of the line, $m=\dfrac{y_2-y_1}{x_2-x_1}$ in this case.

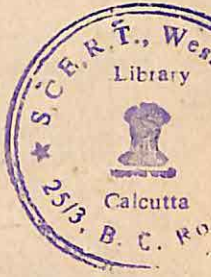
(F) *Perpendicular distance p from the origin on the line, and the angle a which this perpendicular makes with the x-axis are given.*

Let the straight line intersect the axes at A and B, and let p be the perpendicular ON to it from the origin, and α the angle XON made by ON with the x-axis.

Clearly, $OA = ON \sec AON = p \sec \alpha$, and

$OB = ON \sec NOB = p \sec (90^\circ - \alpha) = p \operatorname{cosec} \alpha$.





These being the intercepts on the axes, the equation to the line is

$$\frac{x}{p \sec \alpha} + \frac{y}{p \operatorname{cosec} \alpha} = 1 \text{ or } \mathbf{x \cos \alpha + y \sin \alpha = p}.$$

Note. Here p is always taken with a positive sign. The angle α however may have any value, positive or negative, acute or obtuse. According to the quadrant in which ON lies, the signs of $\cos \alpha$ and $\sin \alpha$ will become definite.

3·2. Reduction of the most general equation $\mathbf{ax+by+c=0}$ of the first degree of a straight line to standard forms.

It may be noted that the most general form of the equation of a straight line, namely $ax + by + c = 0$ can be reduced to any of the standard forms given above.

For instance, the above equation may be written as $ax + by = -c$,

or, $$\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1$$

which is of the form (A), showing that the intercepts on the axes of x and y are $-\dfrac{c}{a}$ and $-\dfrac{c}{b}$ respectively.

Again, we can write the above equation as $by = -ax - c$

or $\quad y = -\frac{a}{b}x - \frac{c}{b}$,

which is of the form (C), showing that its slope on the x-axis, or $m = -\frac{a}{b}$, and its intercept on the y-axis is $-\frac{c}{b}$.

Further, the general equation $ax + by + c = 0$ can be written as $ax + by = -c$, or dividing by $\pm\sqrt{a^2 + b^2}$, we can write it as

$$\frac{a}{\pm\sqrt{a^2+b^2}}x + \frac{b}{\pm\sqrt{a^2+b^2}}y = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

which is of the form (F), as can be evidenced by writing $\cos\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$, whence $\sin\alpha = \frac{b}{\pm\sqrt{a^2+b^2}}$. The perpendicular from the origin on the line, $p = \frac{-c}{\pm\sqrt{a^2+b^2}}$ (where *that sign in the denominator is to be taken which makes the perpendicular p positive in sign*). The sign of the denominator being thus fixed, the angle α is given by

$\cos\alpha = \frac{a}{\pm\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\pm\sqrt{a^2+b^2}}$, which will have their magnitudes and signs definite, and accordingly α will be definitely known, *i.e.*, the quadrant in which the perpendicular from the origin on the line falls is known and p being known, the position of the line is fixed up.

Note. That the intercepts of the line $ax + by + c = 0$ on the axes are $-c/a$ and $-c/b$ can also be obtained by putting $y = 0$ and $x = 0$ respectively in the equation.

3·3. Point of intersection of two given lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

At the intersection point both the equations being satisfied, we get by solving the equations as simultaneous equations, $\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$ giving the co-ordinates of the point of intersection.

3·4. Equation of a straight line through the point of intersection of two given lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$.

(i) The co-ordinates of the point of intersection of the two given lines, by solving the simultaneons equations (as in § *3·3*) are

$$x_1=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\quad y_1=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

Now any line through x_1, y_1 is

$$y-y_1=m(x-x_1)$$

where m is arbitrary and different for different lines.

(ii) The equation $(a_1x+b_1y+c_1)+k(a_2x+b_2y+c_2)=0$ is clearly a first degree equation, and thus represents a straight line. Again, at the intersection point of the two given lines the above equation is satisfied for all values of k, since each of the bracketed portions is zero there. Thus the equation always represents a straight line passing through the intersection of the two given lines. For different values of k, it represents different such lines.

3·5. Condition that three given lines, $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$ may by concurrent.

The point of intersection of the last two lines (by solving them as simultaneous equations) is easily seen to have co-ordinates

$$x=\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2},\quad y=\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}.$$

The first line will pass through this point only if the above co-ordinates satisfy its equation. Hence, the required condition that the three lines may be concurrent is

$$a_1\left(\frac{b_2c_3-b_3c_2}{a_2b_3-a_3b_2}\right)+b_1\left(\frac{c_2a_3-c_3a_2}{a_2b_3-a_3b_2}\right)+c_1=0$$

or, $a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0.$

3·6. Condition that three given points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear.

A, *B*, *C* denoting the given points, if they be collinear, the triangle formed by them is of zero area. For this the condition is [from § *1·4*]

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0.$$

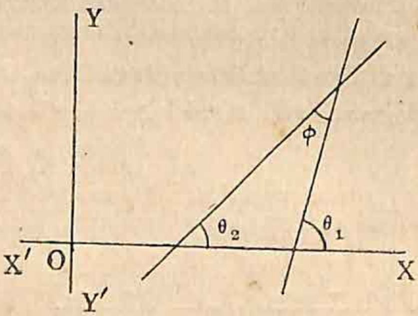
Note. The above condition can also be deduced by writing down the equation of the straight line through any two of the points, and using the condition that this equation is satisfied by the co-ordinates of the third point.

3·7. Angle between two given lines.

(A) *When the lines are given by the equations*

$$y=m_1x+c_1, \quad y=m_2x+c_2.$$

Let θ_1 and θ_2 be the angles made by the given lines with the x-axis.



Then, $\tan\theta_1=m_1$, $\tan\theta_2=m_2$.

Now ϕ being the angle of intersection between the two lines, clearly $\phi=\theta_1 \sim \theta_2$.

$$\therefore \quad \tan\phi=\frac{\tan\theta_1 \sim \tan\theta_2}{1+\tan\theta_1\tan\theta_2}=\frac{m_1 \sim m_2}{1+m_1m_2}.$$

Thus, ϕ is known.

(B) *When the lines are given by the equations*

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

Here the slopes of the lines are clearly given by

$$m_1 = -\frac{a_1}{b_1}, \quad m_2 = -\frac{a_2}{b_2}.$$

Hence, by (A) above,

$$\tan \phi = \frac{\left(-\frac{a_1}{b_1}\right) \sim \left(-\frac{a_2}{b_2}\right)}{1 + \frac{a_1a_2}{b_1b_2}} = \frac{\mathbf{a_1b_2 \sim a_2b_1}}{\mathbf{a_1a_2 + b_1b_2}}.$$

Cor. The two lines will be *parallel* to one another when $\phi = 0$; hence the condition of parallelism is

$$\mathbf{m_1 = m_2} \quad \text{or} \quad \frac{\mathbf{a_1}}{\mathbf{a_2}} = \frac{\mathbf{b_1}}{\mathbf{b_2}}.$$

The two lines will be *perpendicular* to one another when $\phi = \frac{1}{2}\pi$ or $\cot \phi = 0$, giving

$$1 + m_1m_2 = 0 \text{ or } \mathbf{m_1m_2 = -1} \quad \text{or} \quad \mathbf{a_1a_2 + b_1b_2 = 0}$$

as the necessary condition.

Note. From above, it is clear that any straight line parallel to $y = m_1x + c_1$ may have its equation taken as

$y = m_1x + c_2$, where c_2 is arbitrary.

Similarly, any line parallel to $a_1x + b_1y + c_1 = 0$ may have its equation assumed in the form $a_1x + b_1y + c_2 = 0$, where c_2 is different for different such lines.

Again, any line perpendicular to $y = m_1x + c_1$ can be taken to be $y = -\frac{1}{m_1}x + c_2$, and any line perpendicular to $a_1x + b_1y + c_1 = 0$ may be assumed as $b_1x - a_1y + c_2 = 0$, where c_2 is arbitrary in both cases.

These assumptions are very useful in working out examples.

3·8. Illustrative Examples.

Ex. 1. *Find the equation of the straight line passing through the point* $(-5, 3)$ *and perpendicular to the line* $x - 2y + 6 = 0$.

Any line through $(-5, 3)$ has its equation of the form [See § *3·1, D*]

$$y-3=m(x+5) \quad \cdots \quad \text{(i)}$$

The given line is $2y=x+6$ or $y=\frac{1}{2}x+3$, showing that its slope 'm' is $\frac{1}{2}$.

Now, (i) being perpendicular to this line, the necessary condition gives $m\times\frac{1}{2}=-1$ or $m=-2$.

Hence, (i) becomes

$$y-3=-2(x+5), \quad \text{or} \quad 2x+y+7=0$$

which is the equation of the required line.

Ex. 2. *Obtain the equation of the straight line which passes through the intersection of the lines $2x-3y+4=0$ and $3x+4y-5=0$, and has equal intercepts of the same sign along the axes. Find the perpendicular distance from the origin on this straight line.*

Any line passing through the intersection of the two given lines can have its equation written in the form

$$2x-3y+4+k(3x+4y-5)=0,$$

$$\text{or,} \quad (2+3k)x+(4k-3)y+(4-5k)=0 \quad \cdots \quad \text{(i)}$$

Its intercept on the x-axis (by putting $y=0$ in the above equation) is $\dfrac{5k-4}{2+3k}$.

Similarly, (putting $x=0$) the intercept on the y-axis is $\dfrac{5k-4}{4k-3}$.

As the two intercepts are equal in magnitude, and are of the same sign, we have

$$\frac{5k-4}{2+3k}=\frac{5k-4}{4k-3}, \text{ or, } 2+3k=4k-3, \text{ whence } k=5.$$

Substituting in (i), the equation of the required straight line is

$$17x+17y-21=0, \text{ or, } 17x+17y=21 \quad \cdots \quad \text{(ii)}$$

If p denotes the perpendicular distance on it from the origin, and if α be the angle which this perpendicular makes with the positive direction of the x-axis, then the equation of the line can be written as

$$x\cos\alpha+y\sin\alpha=p \qquad \text{[See § } 3{\cdot}1\,(F)\text{]}$$

As this is identical with equation (ii), comparing coefficients, we get

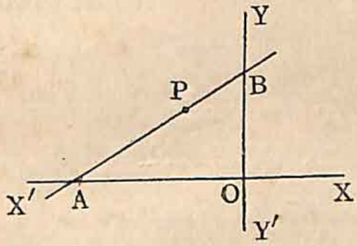
$$\frac{\cos\alpha}{17}=\frac{\sin\alpha}{17}=\frac{p}{21}. \qquad \therefore \cos\alpha=\frac{17p}{21}, \ \sin\alpha=\frac{17p}{21}.$$

$$\therefore \quad \left(\frac{17p}{21}\right)^2+\left(\frac{17p}{21}\right)^2=1 \text{ giving } p^2=\frac{21^2}{17^2+17^2}$$

whence, $p=\dfrac{21}{17\sqrt{2}}=\dfrac{21}{34}\sqrt{2}.$

As the perpendicular from the origin on a line is always taken as positive, the positive sign of the square root is taken.

Ex. 3. *Find the equation to the straight line which passes through the point* $(-4, 3)$ *and is such that the portion of the line between the axes is divided at the point internally in the ratio* 5 : 3.



Let P be the point $(-4, 3)$, and let APB be the line intersecting the axes at A and B, such that $AP : PB = 5 : 3$.

Let the intercepts OA and OB on the axes be a and b respectively in magnitude and sign. Then co-ordinates of A are evidently $(a, 0)$ and those of B are $(0, b)$.

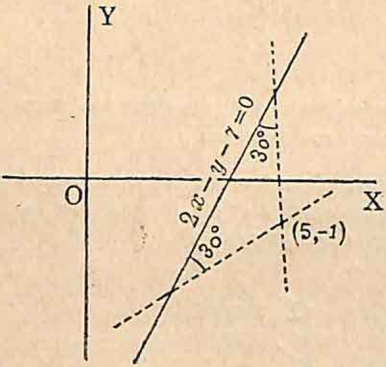
The co-ordinates of the point P which divides AB in the ratio 5 : 3 are then

$$\frac{5.0+3.a}{5+3} \text{ and } \frac{5.b+3.0}{5+3} \ i.e., \ \frac{3a}{8} \text{ and } \frac{5b}{8}.$$

Thus, $\dfrac{3a}{8}=-4$ and $\dfrac{5b}{8}=3$. Hence, $a=-\dfrac{32}{3}$, $b=\dfrac{24}{5}$. The equation to the line AB then, having intercepts $-\dfrac{32}{3}$ and $\dfrac{24}{5}$ along the axes, is

$$\frac{3x}{-32}+\frac{5y}{24}=1, \ i.e., \ 9x-20y+96=0.$$

Ex. 4. *Find the equations to the straight lines passing through the point* $(5, -1)$ *and making an angle of* $30°$ *with the line* $2x-y-7=0$.



Any line passing through $(5, -1)$ is $y+1=m(x-5)$ ··· (i)
The given line

$$y=2x-7$$

has its 'm'$=2$.

If (i) makes an angle $30°$ with it, we have

$$\tan 30° = \frac{2 \sim m}{1+2m}.$$

Thus, $\dfrac{1}{\sqrt{3}}=\dfrac{2-m}{1+2m}$, or $\dfrac{m-2}{1+2m}$.

$\therefore\ m=\dfrac{2\sqrt{3}-1}{2+\sqrt{3}}$, or $-\dfrac{2\sqrt{3}+1}{2-\sqrt{3}}$.

Hence, the equations to the required lines are

$$y+1=\frac{2\sqrt{3}-1}{2+\sqrt{3}}(x-5), \quad \text{and} \quad y+1=-\frac{2\sqrt{3}+1}{2-\sqrt{3}}(x-5).$$

Examples III(a)

1. Obtain the equation to the line inclined to the x-axis at an angle $60°$, and having its intercept on the x-axis equal to -1.

2. Find the equation to the line passing through the point $(-7, 3)$ and having its intercept on the x-axis equal to -5.

3. Find the equation to a straight line passing through the point $(-2, -3)$, and having equal intercepts of opposite sign on the axes.

4. Find the equations to the straight lines joining the pair of points

(i) $(-3, 7)$ and $(-1, -2)$; (ii) $(0, -3)$ and $(7, 0)$;
(iii) $(-2, 6)$ and $(-2, -4)$.

5. Show that the equations of the straight lines joining the points

(i) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$y(t_1 + t_2) - 2x = 2at_1t_2.$$

(ii) $(a \cos \theta, b \sin \theta)$ and $(a \cos \theta', b \sin \theta')$ is

$$\frac{x}{a} \cos \frac{1}{2}(\theta + \theta') + \frac{y}{b} \sin \frac{1}{2}(\theta + \theta') = \cos \frac{1}{2}(\theta - \theta').$$

6. Find the equation to the straight line joining the origin to the point of intersection of the lines $2x - 5y = 1$ and $x + 3y + 7 = 0$.

7. Find the equation to the straight line passing through the point $(-2, 1)$ and also through the intersection of the lines $2x - 3y + 5 = 0$ and $x + 4y - 7 = 0$.

8. (i) Obtain the equation to the straight line passing through the point $(-1, 2)$ and perpendicular to the line $3x + 4y = 5$ [*H. S. 1960, Compartmental*]

(ii) Find the co-ordinates of the foot of the perpendicular from the point $(8, -6)$ on the straight line $2x - 3y + 5 = 0$.

9. (i) Show that the points $(1, 4)$, $(3, -2)$ and $(-3, 16)$ are collinear.

(ii) Prove that the points whose co-ordinates are respectively $(5, 1)$, $(1, -1)$ and $(11, 4)$ lie on a straight line and find the intercepts of this line on the axes.

[*H. S. 1961, Compartmental*]

10. If the points (a, b), (a', b'), $(a - a', b - b')$ are collinear, show that their join passes through the origin, and $ab' = a'b$.

11. (i) Prove that the three lines $2x - 7y + 10 = 0$, $3x - 2y - 2 = 0$ and $x - 7y + 12 = 0$ meet in a point.

(ii) Find for what value of k the three lines $x - y + 5 = 0$, $x + y = 1$ and $y = kx + 13$ will be concurrent.

Find also the point of concurrence.

12. Find the equation to the perpendicular bisector of the straight line joining the poinis $(-2, 7)$ and $(8, -1)$. At what distance is this perpendicular bisector from the origin ? [*H. S. 1961*]

13. Determine the equation of the straight line which passes through the intersection of the lines given by $3x-4y+1=0$ and $5x+y=1$, and has equal intercepts of the same sign along the axes. [*H. S. 1960*]

14. A straight line passes through the intersection of the lines $3x-7y+5=0$, $x-2y-7=0$, and has equal intercepts of the same sign along the axes. Find the length of the perpendicular on it from the origin.

[*H. S. 1961, Compartmental*]

15. A straight line is drawn through the point $(3, 5)$ such that the point bisects the portion of the line intercepted between the axes. Find the equation to the line, and calculate its perpendicular distance from the origin.

[*H. S. 1960, Compartmental*]

16. The portion of a straight line intercepted between the axes is divided internally in the ratio $2:3$ at the point $(-9, -2)$. Show that it passes through the point $(0, -5)$ and is perpendicular to the line $y=3x$.

17. (i) Find the equation of the line joining the point of intersection of the line $x+3y+2=0$, $2x-y-3=0$ to the point of intersection of $7x-y-3=0$, $2x-5y-15=0$.

(ii) Find the equations of the diagonals of the parallelogram formed by the lines $4x-5y-7=0$, $4x-5y-14=0$, $3x+7y-8=0$, $3x+7y-12=0$.

18. Determine the angle between the lines $y=\frac{1}{2}(x-5)$ and $x+3y=2$. Also find out the equation to the line through their point of intersection making a positive angle 60° with the positive direction of the x-axis.

19. Show that the lines $x \cos \alpha + y \sin \alpha = p$,

$x \cos (\alpha + \frac{2}{3}\pi) + y \sin (\alpha + \frac{2}{3}\pi) = p$ and

$x \cos (\alpha - \frac{2}{3}\pi) + y \sin (\alpha - \frac{2}{3}\pi) = p$ form an equilateral triangle.

[*Angle between each pair is* 60°]

20. Find the equations of the two lines through the point $(3, -1)$ which make an angle $45°$ with the line $2x - y = 2$.

21. In any triangle, show analytically that

(i) the medians are concurrent

(ii) the perpendicular bisectors of the sides are concurrent

(iii) the perpendiculars from the vertices on opposite sides are concurrent.

22. Find the ortho-centre of the triangle

(i) whose vertices are $(2, 7)$, $(-6, 1)$ and $(4, -5)$.

(ii) whose sides have equations $2x + 7y + 24 = 0$, $4x + y = 4$, and $x - 3y = 1$.

23. Find the area of the triangle formed by the straight lines whose equations are $x + 2y - 5 = 0$, $y + 2x - 7 = 0$ and $x - y + 1 = 0$.

Find also the co-ordinates of the circum-centre of the triangle.

24. Show that the feet of the perpendiculars from the origin to the lines

$$x + y - 4 = 0,\ x + 5y - 26 = 0,\ 15x - 27y - 424 = 0$$

all lie on a straight line, and find the equation of this line.

25. A line moves so that the sum of the reciprocals of its intercepts on the two axes is constant. Show that it always passes through a fixed point.

ANSWERS

1. $y = \sqrt{3}(x+1)$. **2.** $3x+2y+15=0$. **3.** $x-y=1$.
4. (i) $9x+2y+13=0$. (ii) $3x-7y=21$. (iii) $x=-2$.
6. $32y=15x$. **7.** $8x-23y+39=0$. **8.** (i) $4x-3y+10=0$. (ii) 2, 3.
9. (ii) 3 and $-1\frac{1}{2}$. **11.** (ii) $k=5$; $-2, 3$. **12.** $5x-4y=3$; $\frac{3}{\sqrt{41}}$.

13. $23x+23y=11$. **14.** $85/\sqrt{2}$. **15.** $\frac{x}{6}+\frac{y}{10}=1$; $\frac{15}{17}\sqrt{34}$.

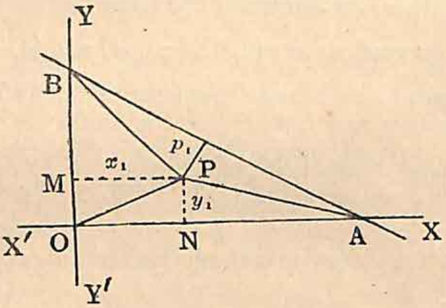
17. (i) $2x-y=3$. (ii) $5x+69y=28$, $37x+29y=112$.

18. $45°$; $5y+3=\sqrt{3}(5x-19)$. **20.** $3x+y=8$, $x-3y=6$.

22. $(-\frac{10}{9}, \frac{49}{27})$; $(\frac{11}{13}, -\frac{7}{13})$. **23.** $\frac{3}{2}$; $(\frac{13}{6}, \frac{11}{6})$. **24.** $3x+y=8$.

3·9. Length of perpendicular from a given point to a given straight line.

(i) *Perpendicular from the point* $P(x_1, y_1)$ *to the straight line* AB *given by the equation* $ax+by+c=0$.



A and B being the points of intersection of the line with the axes, from its equation $ax+by+c=0$, the intercepts OA and OB (putting $y=0$ and $x=0$ respectively) are $-\frac{c}{a}$ and $-\frac{c}{b}$.

$$\therefore\ AB=\sqrt{\left(-\frac{c}{a}\right)^2+\left(-\frac{c}{b}\right)^2}=\pm\frac{c}{ab}\sqrt{a^2+b^2}.$$

PN and PM being the perpendiculars from $P(x_1, y_1)$ on the axes, $PM=x_1$, $PN=y_1$.

Let p_1 be the required perpendicular from P on AB.

First method :

$$\triangle PAB=\triangle OAB-\triangle POA-\triangle POB.$$

$\therefore \quad \frac{1}{2}p_1.AB = \frac{1}{2}OA.OB - \frac{1}{2}PN.OA - \frac{1}{2}PM.OB,$

or, $\quad p_1.\left(\pm\frac{c}{ab}\sqrt{a^2+b^2}\right) = \left(-\frac{c}{a}\right)\left(-\frac{c}{b}\right)$

$$-y_1\left(-\frac{c}{a}\right) - x_1\left(-\frac{c}{b}\right),$$

or, $\quad \pm p_1\sqrt{a^2+b^2} = ax_1 + by_1 + c.$

$$\therefore \quad \mathbf{p_1 = \frac{ax_1 + by_1 + c}{\pm\sqrt{a^2+b^2}}}.$$

Second method :

P, A, B have co-ordinates (x_1, y_1), $\left(-\frac{c}{a}, 0\right)$ and $\left(0, -\frac{c}{b}\right)$ respectively.

$$\therefore \triangle PAB = \frac{1}{2}\left\{x_1\left\{0+\frac{c}{b}\right) + \left(-\frac{c}{a}\right)\left(-\frac{c}{b} - y_1\right) + 0(y_1 - 0)\right\}$$

$$\therefore \quad \tfrac{1}{2}p_1.AB = \frac{1}{2}\left(\frac{cx_1}{b} + \frac{c^2}{ab} + \frac{cy_1}{a}\right),$$

or, $\quad p_1\left(\pm\frac{c}{ab}\sqrt{a^2+b^2}\right) = \frac{c(ax_1 + by_1 + c)}{ab}.$

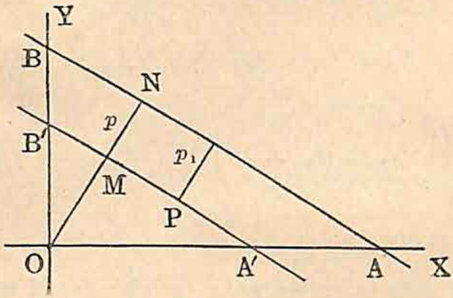
$$\therefore \quad p_1 = \frac{ax_1 + by_1 + c}{\pm\sqrt{a^2+b^2}}.$$

Note. To choose the proper sign of the denominator, it should be noted that the *perpendicular from the origin* (0, 0) to the line *is always taken as positive.*

$\therefore$ the *sign of the denominator should be taken same as that of* c, as is apparent by putting $x_1 = 0$, $y_1 = 0$.

The perpendicular distance of the point P from the line AB is thus positive, if P lies on the same side as the origin with respect to the line AB. For this case $ax_1 + by_1 + c$ has the same sign as that of c. If P be on the opposite side of the origin with respect to the line AB, the perpendicular from P on AB will be negative, and in this case, $ax_1 + by_1 + c$ will have its sign opposite to that of c. In this connection, see § *3·10*.

(ii) *Perpendicular from the point* $P(x_1, y_1)$ *to the line* $x \cos a + y \sin a = p$.



Here, from the equation to the line, the perpendicular from the origin on it is $ON = p$, and ON makes an angle a with the x-axis.

Let p_1 be the perpendicular from the given point $P(x_1, y_1)$ to the given line AB. If $A'B'$ denotes the parallel line through P, clearly ON is perpendicular to $A'B'$ as well, and the perpendicular OM from O on $A'B'$ is clearly $p - p_1$.

Hence, equation to the line $A'B'$ is

$$x \cos a + y \sin a = p - p_1.$$

As it passes through $P(x_1, y_1)$, we must have

$$x_1 \cos a + y_1 \sin a = p - p_1.$$

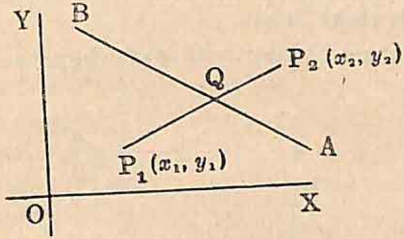
$$\therefore \quad \mathbf{p_1 = p - x_1 \cos \alpha - y_1 \sin \alpha.}$$

Note. Case (i) can be deduced as a corollary to (ii), for the equation to the line being of the form $ax + by + c = 0$, it is reduced to the form of (ii) by dividing by $\pm \sqrt{a^2 + b^2}$ [See § 3·2]. Thus $\dfrac{ax + by + c}{\pm \sqrt{a^2 + b^2}}$ (when proper sign is selected) is of the form $p - x \cos a - y \sin a$, p being positive and $\dfrac{c}{\pm \sqrt{a^2 + b^2}}$ also positive by proper choice of sign.

3·10. Theorem.

The points (x_1, y_1) *and* (x_2, y_2) *are on the same side or opposite sides of the straight line* $ax + by + c = 0$, *according*

as the expressions ax_1+by_1+c *and* ax_2+by_2+c *are of the same or opposite signs.*



Let P_1 and P_2 be the points (x_1, y_1) and (x_2, y_2) and AB the straight line $ax+by+c=0$. ... ... (i)

Let the line P_1P_2 intersect AB at Q where the ratio $P_1Q : QP_2$ is $\lambda : 1$. Then the co-ordinates of Q are

$$\frac{\lambda x_2+x_1}{\lambda+1},\quad \frac{\lambda y_2+y_1}{\lambda+1}.$$

As Q lies on AB, these co-ordinates must satisfy the equation (i).

Hence, $a\dfrac{\lambda x_2+x_1}{\lambda+1}+b\dfrac{\lambda y_2+y_1}{\lambda+1}+c=0$,

or, $\lambda(ax_2+by_2+c)+(ax_1+by_1+c)=0$;

$\therefore\quad \lambda=-\dfrac{ax_1+by_1+c}{ax_2+by_2+c}$.

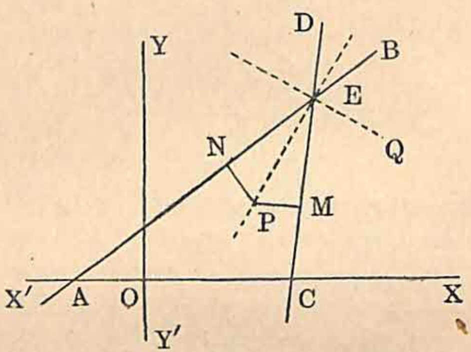
If P_1 and P_2 be on opposite sides of AB, Q must be within P_1P_2 and so the ratio $\lambda : 1$ is positive ; if P_1 and P_2 be on the same side of AB, Q must be external to AB and so $\lambda : 1$ is negative.

Hence, P_1 and P_2 will be on the same or opposite sides of AB according as $-\dfrac{ax_1+by_1+c}{ax_2+by_2+c}$ is negative or positive, *i.e.*, as ax_1+by_1+c and ax_2+by_2+c are of the same or opposite signs.

Cor. The point $P_1(x_1, y_1)$ is on the same side of the line $ax+by+c=0$ as the origin if ax_1+by_1+c is of the same sign as of c.

3·11. Equations of bisectors of the angles between two given straight lines

$$\mathbf{a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0.}$$



Any point x_1, y_1 (say) on any of the bisectors must be equidistant from the two lines, so that the perpendicular distances from it to the two lines must be equal in magnitude. If it be on the bisector of the angle in which the origin lies (as at P) both perpendiculars must be of the sams sign, because the point and the origin lie on the same side with respect to each of the lines. On the other hand, if it be on the bisector of the angle in which the origin does not lie (as at Q) clearly it must be on the same side of the origin with respect to one of the lines (AB), and on opposite side of the origin with respect to the other line (CD). Hence, the two perpendiculars, being equal in magnitude, must be opposite in sign.

Hence, for points on one of the bisectors,

$$\frac{a_1x_1+b_1y_1+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x_1+b_2y_1+c_2}{\sqrt{a_2^2+b_2^2}}$$

and for points on the other,

$$\frac{a_1x_1+b_1y_1+c_1}{\sqrt{a_1^2+b_1^2}}=-\frac{a_2x_1+b_2y_1+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Hence, the equations to the bisectors of the angles between the two given lines (*i.e.*, the locus of x_1, y_1) are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}.$$

Note. If c_1 and c_2 be of the same sign, the + sign on the right-hand side will give the bisector of the angle in which the origin lies, and − sign will give the other bisector.

If c_1 and c_2 be of opposite signs, the − sign on the right-hand side will give the bisector of the angle containing the origin, and + sign will give the other bisector.

3·12. Illustrative Examples.

Ex. 1. *Find the perpendicular distance from the point* $-2, -7$ *to the straight line* $7y-24x=10$. *Is the point on the same side of the line as the origin, or on the opposite side?*

The equation of the line can be written as

$$24x-7y+10=0.$$

The perpendicular distance from $(-2, -7)$ on it is given by

$$p=\frac{24.(-2)-7(-7)+10}{\pm\sqrt{24^2+7^2}}.$$

The proper sign to be chosen for the denominator here is + (same as the sign of the constant term +10).

Hence, $$p=\frac{-48+49+10}{+25}=\frac{11}{25}.$$

As $24(-2)-7(-7)+10$ (*i.e.*, +11) has the same sign as of the constant term +10, the point $(-2, -7)$ is on the same side of the given line as the origin.

The perpendicular from the point on the line is thus to be associated with positive sign, as has been found to be the case.

Ex. 2. *Find the bisector of the angle containing the origin between the lines* $4x-3y+5=0$ *and* $5x-12y-2=0$.

The bisectors of the angles between the given lines are [See § *3·11*],

$$\frac{4x-3y+5}{\sqrt{4^2+3^2}}=\pm\frac{5x-12y-2}{\sqrt{5^2+12^2}}.$$

Of these, (by *note of* § *3·11*), the $-$sign on the right-hand will give the bisector of the angle containing the origin ($\because$ here 5 and -2 are of opposite signs).

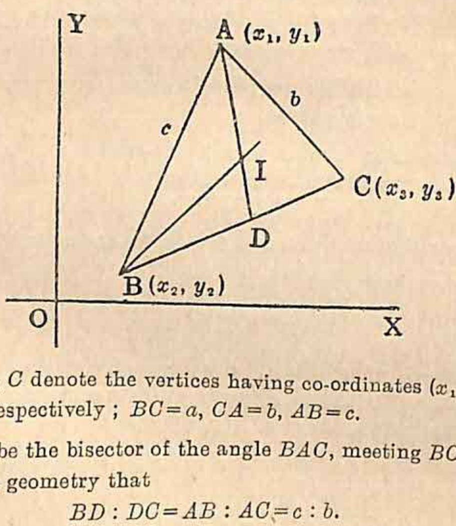
Thus, the required bisector is

$$\frac{4x-3y+5}{5}=-\frac{5x-12y-2}{13},$$

or, $77x-99y+55=0$, *i.e.*, $7x-9y+5=0$.

Ex. 3. *If* (x_1, y_1), (x_2, y_2) *and* (x_3, y_3) *be the co-ordinates of the vertices of a triangle, and* a, b, c *respectively be the lengths of the sides opposite to these vertices, then the co-ordinates of the in-centre of the triangle are*

$$\frac{ax_1+bx_2+cx_3}{a+b+c} \text{ and } \frac{ay_1+by_2+cy_3}{a+b+c}.$$



Let A, B, C denote the vertices having co-ordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively ; $BC=a$, $CA=b$, $AB=c$.

Let AD be the bisector of the angle BAC, meeting BC at D ; then we know from geometry that

$$BD : DC = AB : AC = c : b.$$

$\therefore$ the co-ordinates of D are [See § *1·3*]

$$\frac{cx_3+bx_2}{c+b},\ \frac{cy_3+by_2}{c+b}.$$

Also, $\because\ \dfrac{BD}{DC}=\dfrac{c}{b}$, $\therefore\ \dfrac{BD}{BC}=\dfrac{c}{b+c}$. Hence, $BD=\dfrac{ac}{b+c}$.

Now, BI being the bisector of the angle ABC, meeting AD at I, I is the in-centre of the triangle ABC.

Also, $\frac{DI}{IA}=\frac{BD}{BA}=\frac{ac}{b+c}:c=a:(b+c)$.

Thus, the co-ordinates of D and A being known, the co-ordinates of I are

$$\frac{a.x_1+(b+c)\cdot\frac{cx_3+bx_2}{c+b}}{a+(b+c)} \text{ and } \frac{a.y_1+(b+c)\cdot\frac{cy_3+by_2}{b+c}}{a+(b+c)}$$

$$i.e., \quad \frac{ax_1+bx_2+cx_3}{a+b+c},\ \frac{ay_1+by_2+cy_3}{a+b+c}.$$

Examples III(b)

1. Find the perpendicular distance (i) from the point $(-1, -3)$ to the straight line $5x-12y+8=0$, (ii) from the point (a, b) to the straight line $\frac{x}{a}-\frac{y}{b}=1$.

2. Show that the distance of the point $(-5, 2)$ from the point $(-1, -1)$ is equal to its perpendicular distance from the straight line $4(y-1)=3(x-2)$.

3. Show that the point $(-3, 1)$ is equidistant from the two lines $4x-3y=2$ and $24x+7y=20$.

4. Find the perpendicular distance of the point of intersection of the two lines $2x+3y+6=0$ and $5x-y-19=0$ from the line $6x-8y=0$. What are the co-ordinates of the foot of the perpendicular ?

5. Find the points on the y-axis whose perpendicular distance from the straight line $5y=12x-16$ is 2.

6. Find the distance between the parallel lines $3x-4y+7=0$ and $3(x-2)=4y-3$.

7. (i) Show that the product of the perpendiculars from the two points $(\pm 4, 0)$ on the straight line $3x\cos\theta+5y\sin\theta=15$ is independent of θ.

(ii) Show that the product of the perpendiculars drawn from the two points $(\pm\sqrt{a^2-b^2}, 0)$ upon the straight line

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \text{ is } b^2.$$

8. (i) Find the perpendicular distance of the point (2, 2) from the line joining the points (– 5, 1) and (3, – 5).

(ii) Prove that the points (2, – 6) and (– 4, – 12) are equidistant from the line joining (1, – 4) and (– 3, – 14), and are on opposite sides of it.

9. Show that the points (2, – 3) and (– 4, – 2) are on opposite sides of the line $5x-7y-11=0$. Which of the points is on the same side of the line as the origin ?

10. Prove that the origin is inside the triangle whose vertices are (2, 1), (3, – 2), (– 4, – 1).

[*The origin lies on the same side with each vertex with respect to the line joining the other two.*]

11. Find the bisector of the angle between the lines $12x+5y=4$ and $3x+4y+7=0$ containing the origin.

12. Find the bisectors of the angles between the lines $24x-7y-2=0$ and $3y-4x+7=0$, and verify that these bisectors are at right angles. Which of these bisect the angle in which the origin lies ?

13. Find the in-centre of the triangle

(i) whose vertices are (– 1, – 2), (– 1, 3) and (11, – 2).

(ii) whose sides are $x=3$, $y=-4$ and $3x+4y=17$.

14. The algebraic sum of the perpendicular distances on a straight line from the three given points (5, – 3), (– 2, 4) and (– 3, – 7) is zero. Show that the straight line passes through a fixed point.

ANSWERS

1. (i) 3. (ii) $\dfrac{ab}{\sqrt{a^2+b^2}}$. **4.** 5 ; (0, 0). **5.** (0, 2) and $\left(0, -\dfrac{42}{5}\right)$.

6. 2. **8.** (i) 5. **9.** (– 4, – 2). **11.** $99x+77y+71=0$.

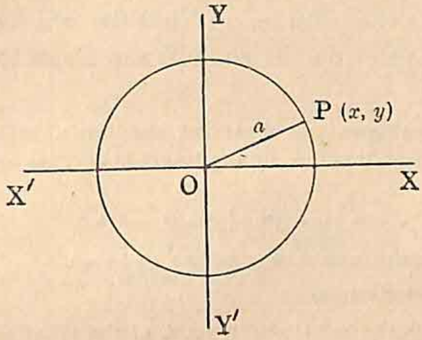
12. $44x-22y-37=0$, $4x+8y+33=0$; latter.

13. (i) (1, 0). (ii) (5, – 2).

CHAPTER IV

CIRCLE

4·1. Circle of given radius, with centre at the origin.

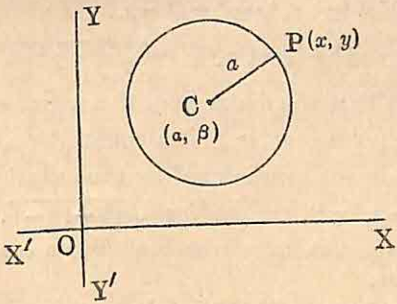


Let a be the radius of a circle whose centre is at the origin O. If P (x, y) be any point on the circle, the distance $OP = a$,

or, $OP^2 = a^2$; $\quad \therefore \; \mathbf{x^2 + y^2 = a^2}$.

This being the relation satisfied by the co-ordinates (x, y) of any point on the circle, it represents the equation to the circle.

4·2. Circle of given radius, with centre at any given point.



Let $C\ (\alpha, \beta)$ be the centre, and a the radius of a circle. If $P\ (x, y)$ be any point on the circle, the distance $CP = a$, or $CP^2 = a^2$; $\therefore\ (\mathbf{x} - \alpha)^2 + (\mathbf{y} - \beta)^2 = \mathbf{a}^2$.

This being the relation satisfied by the co-ordinates (x, y) of any point on the circle, it represents the equation to the circle.

Note. From above it is clear that any circle, with centre any where (say α, β) and radius of any length (a say) has equation of the form

$$x^2 + y^2 - 2\alpha x - 2\beta y + (\alpha^2 + \beta^2 - a^2) = 0,$$

i.e., of the form $x^2 + y^2 + 2gx + 2fy + c = 0$

where g, f, c are constants.

This is thus the most general form of the equation to a circle. [see also § *4·3* and its note]

For a circle with centre as origin, g and f are zeroes.

4·3. *To show that the equation*

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$$

always represents a circle (for all values of the constants g, f, c), and to find its centre and radius.

The given equation can be written in the form

$$x^2 + 2gx + g^2 + y^2 + 2fy + f^2 = g^2 + f^2 - c,$$

or, $(x + g)^2 + (y + f)^2 = g^2 + f^2 - c,$

or, $\{x - (-g)\}^2 + \{y - (-f)\}^2 = (\sqrt{g^2 + f^2 - c})^2.$

This shows that the distance of the variable point (x, y) from the fixed point $(-g, -f)$ is a constant $= \sqrt{g^2 + f^2 - c}$.

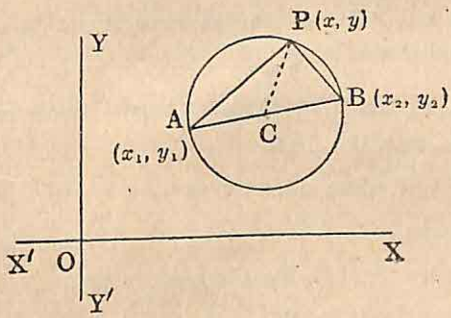
Hence, the locus represented by the equation is a circle, with *centre* $(-\mathbf{g}, -\mathbf{f})$, and *radius* $\sqrt{\mathbf{g}^2 + \mathbf{f}^2 - \mathbf{c}}$.

Note. The equation may be multiplied by a constant factor a, and put in the form

$$ax^2 + ay^2 + 2g'x + 2f'y + c' = 0 \quad \ldots \quad \ldots \quad \ldots \quad \text{(i)}$$

This also represents a circle, but the centre here is not $(-g', -f')$, nor is the radius $= \sqrt{g'^2+f'^2-c'}$. In fact, whenever in a second degree equation (in rectangular co-ordinates) the coefficients of x^2 and y^2 are equal, and there is no term involving the product xy, the equation represents a circle. The equation (i) is thus the most general form of the equation to a circle. To get its centre and radius, we are to divide out the equation by the common coefficient a of x^2 or y^2, and reduce it to the form $x^2+y^2+2gx+2fy+c=0$. Then $(-g, -f)$ are the co-ordinates of the centre, and $\sqrt{g^2+f^2-c}$ is the length of the radius.

4·4. Circle with the given points x_1, y_1 and x_2, y_2 as extremities of a diameter.



$A(x_1, y_1)$ and $B(x_2, y_2)$ being the extremities of a diameter, the mid-point of AB, whose co-ordinates are $\frac{1}{2}(x_1+x_2)$, $\frac{1}{2}(y_1+y_2)$ is the centre. Also the radius $=\frac{1}{2}AB=\frac{1}{2}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Hence, the equation to the circle is

$$\{x-\tfrac{1}{2}(x_1+x_2)\}^2+\{y-\tfrac{1}{2}(y_1+y_2)\}^2 = \tfrac{1}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\} \quad \cdots \quad \text{(i)}$$

Alternatively

P being any point (x, y) on the circle, AB being a diameter, PA and PB must be at right angles. Now 'm' of PA is $\frac{y-y_1}{x-x_1}$ and that of PB is $\frac{y-y_2}{x-x_2}$. [See § 3·1(E)]

Hence, for PA and PB to be at right angles,

$$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1,$$

or, $$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0} \quad \cdots \quad \text{(ii)}$$

which being the equation satisfied by the co-ordinates of any point P on the circle, it represents the required equation to the circle.

Note. It may be noted that the forms (i) and (ii) are identical, as can be shown by simplifying.

4·5. Circle passing through any three given points (x_1, y_1), (x_2, y_2), (x_3, y_3).

Let $$x^2+y^2+2gx+2fy+c=0 \quad \cdots \quad \cdots \quad \text{(i)}$$

be the equation to the circle.

As it passes through the three given points,

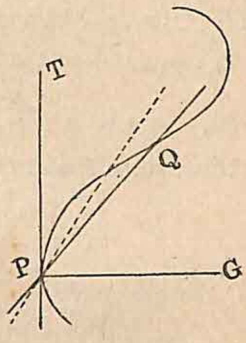
$$x_1^2+y_1^2+2gx_1+2fy_1+c=0$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0$$
$$x_3^2+y_3^2+2gx_3+2fy_3+c=0.$$

From these three equations, which are linear equations in the unknowns g, f, c, we get definite values of these unknowns when (x_1, y_1), (x_2, y_2) and (x_3, y_3) are given.

Substituting these values of g, f, c in (i), we get the required equation of the circle, as also its centre $(-g, -f)$ and radius $\sqrt{g^2+f^2-c}$.

4·6. Definitions of Tangent and Normal.

If P be any point on a curve, and if we take a neighbouring point Q on it, then if the line joining P and Q be turned about P so that the other point of intersection Q gradually approaches P, the limiting position PT of the line PQ, when Q ultimately coincides with P, is defined as the *tangent* to the curve at P.

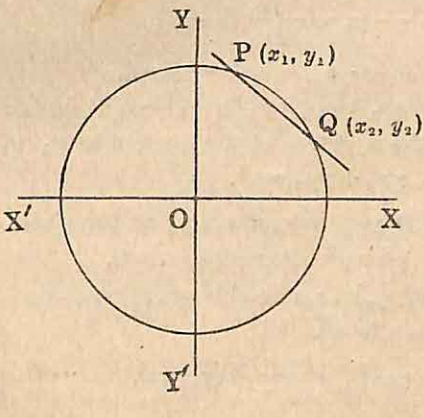


The line PG through P, perpendicular to the tangent at P, is defined as the *normal* to the curve at P.

4·7. Equation to the tangent at a given point x_1, y_1 on the circle :

(A) $x^2+y^2=a^2$.

(B) $x^2+y^2+2gx+2fy+c=0$.



(A) Let P be the point (x_1, y_1) on the circle

$$x^2+y^2=a^2, \qquad \cdots \quad \text{(i)}$$

and let $Q(x_2, y_2)$ be a neighbouring point on it.

The equation to the chord PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \cdots \quad \text{(ii)}$$

Now, since P and Q both lie on the circle (i), we have

$$x_1^2+y_1^2=a^2 \qquad \cdots \qquad \cdots \text{(iii)}$$

$$x_2^2+y_2^2=a^2 \qquad \cdots \qquad \cdots \text{(iv)}$$

$\therefore$ subtracting,

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)=0,$$

whence $\dfrac{y_2-y_1}{x_2-x_1}=-\dfrac{x_2+x_1}{y_2+y_1}$.

$\therefore$ equation (ii) can be written as

$$y-y_1=-\frac{x_2+x_1}{y_2+y_1}(x-x_1). \qquad \cdots \text{(v)}$$

Now make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position, the chord PQ becomes the tangent at P, whose equation, [*from (v)*] then becomes

$$y-y_1=-\frac{2x_1}{2y_1}(x-x_1),$$

or, $\quad x_1(x-x_1)+y_1(y-y_1)=0,$

i.e., $\quad xx_1+yy_1=x_1^2+y_1^2=a^2$ [*by (iii)*].

Hence, the equation of the tangent at x_1, y_1 is

$$\mathbf{xx_1+yy_1=a^2}.$$

(B) Let P be the point x_1, y_1 on the circle

$$x^2+y^2+2gx+2fy+c=0, \qquad \cdots \quad \text{(i)}$$

and let $Q(x_2, y_2)$ be a neighbouring point on it. The equation to the chord PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

Now, since P and Q both lie on the circle (i), we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \cdots \text{(iii)}$$

and $$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad \cdots \text{(iv)}$$

$\therefore$ subtracting,

$$(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0,$$

whence, $(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0$,

or, $$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}.$$

Hence, equation (ii) of chord PQ can be written as

$$y - y_1 = -\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}(x - x_1).$$

Now making Q approach P and ultimately coincide with it (so that x_2, y_2 coincide with x_1, y_1), the chord becomes the tangent at P, whose equation is then

$$y - y_1 = -\frac{2(x_1 + g)}{2(y_1 + f)}(x - x_1),$$

or, $$(x_1 + g)(x - x_1) + (y_1 + f)(y - y_1) = 0,$$

or, $$xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1,$$

or, adding $gx_1 + fy_1 + c$ to both sides,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0. \quad [\textit{by (iii)}]$$

Hence, the tangent to the circle (i) at x_1, y_1 is

$$\mathbf{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.}$$

4·8. Equation to the normal at $\mathbf{x_1, y_1}$ to the circle :

(A) $\mathbf{x^2 + y^2 = a^2}$.

(B) $\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}$.

(A) The tangent at x_1, y_1 to the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$, or, $y=-\frac{x_1}{y_1}x+\frac{a^2}{y_1}$, of which the '$m$' is $-\frac{x_1}{y_1}$.

The normal, which is perpendicular to the tangent, through x_1, y_1 is then

$$y-y_1=\frac{y_1}{x_1}(x-x_1),$$

$$\text{or,}\quad \frac{x-x_1}{x_1}=\frac{y-y_1}{y_1},\quad \text{or,}\quad \mathbf{\frac{x}{x_1}=\frac{y}{y_1}},$$

which evidently passes through the origin, *i.e.*, the centre of the circle.

(B) The tangent at x_1, y_1 to the circle

$$x^2+y^2+2gx+2fy+c=0,$$

$$\text{is}\qquad xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0,$$

$$\textit{i.e.,}\qquad x(x_1+g)+y(y_1+f)+(gx_1+fy_1+c)=0$$

of which the 'm' is $-\frac{x_1+g}{y_1+f}$.

The normal, which is perpendicular to the tangent, through x_1, y_1 is then

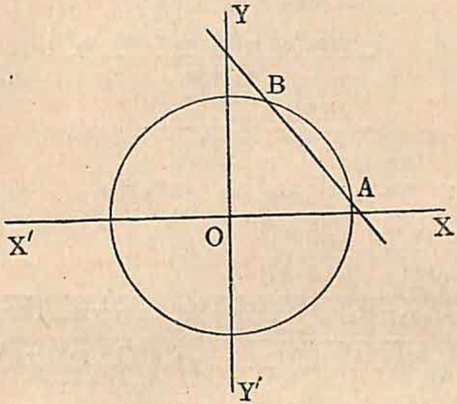
$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1),$$

$$\text{or,}\qquad \mathbf{x(y_1+f)-y(x_1+g)=fx_1-gy_1}.$$

Note. Evidently this line passes through the point $(-g, -f)$, which is the centre of the circle.

Thus, *the normal at any point of a circle passes through its centre, in other words, the radius to any point of a circle is perpendicular to the tangent at the point.*

4·9. Length of the chord of the circle $x^2+y^2=a^2$, intercepted by the straight line $y=mx+c$.



At the points of intersection of the line with the circle, both the equations are satisfied. Hence, eliminating y between the two equations, the abscissæ of the points of intersection will be given by

$$x^2+(mx+c)^2=a^2,$$

$$\text{or,}\quad x^2(1+m^2)+2mcx+(c^2-a^2)=0, \qquad \cdots \quad \text{(i)}$$

which being a quadratic equation in x, there are only two values of x and accordingly only two points of intersection of the straight line with the circle (which may be real and distinct, real and coincident, or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points A and B of intersection. Then x_1 and x_2 are the roots of (i).

$$\therefore\quad x_1+x_2=-\frac{2mc}{1+m^2} \text{ and } x_1x_2=\frac{c^2-a^2}{1+m^2}.$$

$$\therefore\quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$$
$$=\frac{4m^2c^2}{(1+m^2)^2}-\frac{4(c^2-a^2)}{1+m^2}$$
$$=\frac{4\{m^2c^2-(c^2-a^2)(1+m^2)\}}{(1+m^2)^2}$$
$$=\frac{4\{a^2(1+m^2)-c^2\}}{(1+m^2)^2}.$$

Again, $y_1=mx_1+c$ and $y_2=mx_2+c$.

$\therefore\quad y_1-y_2=m(x_1-x_2)$.

Length of the chord AB

$$=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(x_1-x_2)^2(1+m^2)}$$
$$=\sqrt{\frac{4\{a^2(1+m^2)-c^2\}}{1+m^2}}=2\frac{\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}.$$

Cor. Condition of tangency.

The given line will touch the circle if the chord intercepted is zero. Hence the condition that the given line $y=mx+c$ may touch the circle $x^2+y^2=a^2$ is $c^2=a^2(1+m^2)$,

$$\text{or,}\quad \mathbf{c=\pm a\sqrt{1+m^2}.}$$

Alternative method for condition of tangency.

The perpendicular from the centre of the circle to the line is equal to the radius.

$\therefore$ perpendicular from 0, 0 on $mx-y+c=0$ is equal to a,

$$\text{or,}\quad \frac{c}{\pm\sqrt{1+m^2}}=a;$$
$$\therefore\quad c=\pm a\sqrt{1+m^2}.$$

Note. There are thus two tangents to the circle parallel to a given line $y=mx+c$, (*i.e.*, with a given m), namely

$$y=mx\pm a\sqrt{1+m^2}.$$

4·10. *To show that $y=mx+a\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=a^2$, and to find the point of contact.*

The tangent at x_1, y_1 of the circle is

$$xx_1 + yy_1 = a^2 \quad \text{or} \quad xx_1 + yy_1 - a^2 = 0 \quad \cdots \quad \text{(i)}$$

If the line $y = mx + a\sqrt{1+m^2}$ or $mx - y + a\sqrt{1+m^2} = 0$ (ii) be a tangent to the circle at (x_1, y_1), the equations (i) and (ii) must be the same. Hence, comparing coefficients,

$$\frac{x_1}{m} = \frac{y_1}{-1} = \frac{-a^2}{a\sqrt{1+m^2}} = \frac{-a}{\sqrt{1+m^2}}.$$

$$\therefore \qquad x_1 = -\frac{am}{\sqrt{1+m^2}} \quad y_1 = \frac{a}{\sqrt{1+m^2}}.$$

The line (ii) therefore will touch the circle only if the assumed point x_1, y_1 is really a point on the circle,

$$i.e., \text{ if } \left(\frac{-am}{\sqrt{1+m^2}}\right)^2 + \left(\frac{a}{\sqrt{1+m^2}}\right)^2 = a^2$$

which is evidently satisfied.

Thus, $y = mx + a\sqrt{1+m^2}$ touches the circle, whatever m may be and the point of contact is given by

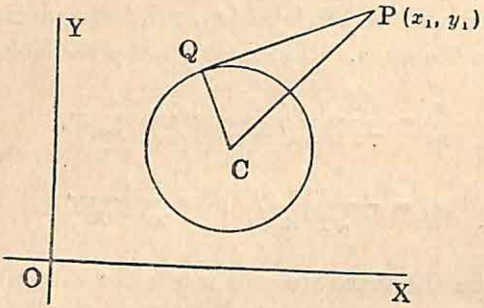
$$\mathbf{x_1 = -\frac{am}{\sqrt{1+m^2}}, \; y_1 = \frac{a}{\sqrt{1+m^2}}}.$$

Note. Similarly, $y = mx - a\sqrt{1+m^2}$ is also a tangent to the circle $x^2 + y^2 = a^2$, the point of contact being $\frac{am}{\sqrt{1+m^2}}, \frac{-a}{\sqrt{1+m^2}}$.

4·11. *Length of the tangent from an external point* x_1, y_1 *to the circle* $x^2 + y^2 + 2gx + 2fy + c = 0$.

Let P be the point (x_1, y_1) and PQ a tangent to the circle from P. The centre C of the circle is $(-g, -f)$, and the radius CQ is $\sqrt{g^2 + f^2 - c}$.

Now since, as proved before (§ 4·8, *Note*), CQ is perpendicular to CP,



$$PQ^2 = CP^2 - CQ^2$$
$$= (x_1+g)^2 + (y_1+f)^2 - (g^2+f^2-c)$$
$$= x_1^{\ 2} + y_1^{\ 2} + 2gx_1 + 2fy_1 + c.$$

$$\therefore \quad \mathbf{PQ = \sqrt{x_1^{\ 2}+y_1^{\ 2}+2gx_1+2fy_1+c}.}$$

Cor. The length of the tangent from the external point (x_1, y_1) to the circle $x^2+y^2=a^2$ is $\sqrt{x_1^{\ 2}+y_1^{\ 2}-a^2}$.

Note. We notice that when the co-ordinates x_1, y_1 of any point are substituted for x, y on the left-hand side of the equation to a circle $x^2+y^2+2gx+2fy+c=0$ or $x^2+y^2-a^2=0$, we obtain the square on the length of the tangent from the point x_1, y_1 to the circle. If this is positive, the point is outside the circle, and the length of the tangent is real. If the expression is negative, the point is inside the circle, and the length of the tangent is imaginary.

If the equation to the circle be in the form $ax^2+ay^2+2g'x+2f'y+c'=0$, we are first to divide out the equation by a, and reduce it to the form $x^2+y^2+2gx+2fy+c=0$ and then substitute (x_1, y_1) on the left side to get the square on the length of the tangent. [See § 4·3 *note* in this connection.]

4·12. Illustrative Examples.

Ex. 1. *Find the equation to the circle passing through the points* $(2, -3)$ *and* $(-3, -4)$ *and having its centre on the line* $7x+2y+6=0$.

Let α, β be the co-ordinates of the centre of the circle. As the circle passes through the points $(2, -3)$ and $(-3, -4)$, these points must be equidistant from the centre. Hence,

$$(\alpha-2)^2+(\beta+3)^2=(\alpha+3)^2+(\beta+4)^2,$$

$$\text{or,}\quad 10\alpha+2\beta+12=0,\ \textit{i.e.},\ 5\alpha+\beta+6=0. \qquad \cdots \quad \text{(i)}$$

Also, since the centre lies on the given line, we have

$$7\alpha+2\beta+6=0. \qquad \cdots \qquad \cdots \qquad \cdots \text{(ii)}$$

From (i) and (ii), solving, $\alpha=-2$, $\beta=4$. The radius of the circle, r, is equal to the distance of the point $2, -3$ from the centre $-2, 4$.

$$\therefore\quad r^2=(2+2)^2+(-3\ -4)^2=65.$$

Hence, the required equation to the circle is

$$(x-\alpha)^2+(y-\beta)^2=r^2, \quad \text{or,}\quad (x+2)^2+(y-4)^2=65,$$

$$\text{or,}\quad x^2+y^2+4x-8y-45=0.$$

Ex. 2. *Find the length of the chord intercepted by the straight line* $3x-4y+5=0$ *of the circle passing through the points* $(1, 2)$, $(3, -4)$ and $(5, -6)$.

$$\text{Let}\quad x^2+y^2+2gx+2fy+c=0 \qquad \cdots \qquad \cdots \qquad \cdots \quad \text{(i)}$$

be the equation to the circle passing through the given points $(1, 2)$, $(3, -4)$ and $(5, -6)$.

Then, substituting values of the co-ordinates,

$$5+2g+4f+c=0$$

$$25+6g-8f+c=0$$

$$\text{and}\quad 61+10g-12f+c=0.$$

Solving these, we get $g=-11$, $f=-2$, $c=25$.

Hence, the circle (i) becomes

$$x^2+y^2-22x-4y+25=0 \qquad \cdots \qquad \cdots \quad \text{(ii)}$$

The given straight line is

$$3x-4y+5=0 \qquad \cdots \qquad \cdots \qquad \cdots \text{(iii)}$$

For the common points of intersection of (ii) and (iii), eliminating y, the abscissæ are the roots of

$$x^2+\left(\frac{3x+5}{4}\right)^2-22x-(3x+5)+25=0,$$

$$\text{or,}\quad 5x^2-74x+69=0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(iv)}$$

If (x_1, y_1) and (x_2, y_2) be the co-ordinates of the points of intersection of (ii) and (iii), then x_1 and x_2 are roots of (iv).

$$\therefore\quad x_1+x_2=\tfrac{74}{5},\ x_1x_2=\tfrac{69}{5}.$$

$$\therefore\quad (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=(\tfrac{74}{5})^2-4.\tfrac{69}{5}=\tfrac{4096}{25}.$$

Also, (x_1, y_1) and (x_2, y_2) both lying on (iii),

$$3x_1-4y_1+5=0,\ 3x_2-4y_2+5=0$$

$$\therefore\quad 3(x_1-x_2)-4(y_1-y_2)=0. \quad \therefore\quad (y_1-y_2)^2=\tfrac{9}{16}(x_1-x_2)^2.$$

$\therefore$ the length of the intercepted chord being l,

$$l^2=(x_1-x_2)^2+(y_1-y_2)^2=(1+\tfrac{9}{16})(x_1-x_2)^2$$
$$=\tfrac{25}{16}\times\tfrac{4096}{25}=256.$$

$$\therefore\quad l=16.$$

Alternatively

The centre of the circle (ii) is 11, 2 and its radius r is $\sqrt{11^2+2^2-25}=10$ [See § *4·3*].

Perpendicular from this centre on line (iii) is

$$p=\frac{3.11-4.2+5}{\sqrt{3^2+4^2}}=6.$$

Now if AB be the chord along the line, and CN be the perpendicular on AB from the centre C, then N is the mid-point of AB. Also $AN^2=CA^2-CN^2$.

Hence, the length of the chord intercepted is

$$AB=2AN=2\sqrt{r^2-p^2}=2\sqrt{100-36}=16.$$

Ex. 3. *Show that the straight line* $4x+3y-31=0$ *touches the circle* $x^2+y^2-6x+4y=12$, *and find the point of contact.*

If possible, let (x_1, y_1) be the co-ordinates of the point on the circle

$$x^2+y^2-6x+4y-12=0 \quad \cdots \quad \cdots \quad \text{(i)}$$

at which the given line

$$4x+3y-31=0 \quad \cdots \quad \cdots \quad \cdots \quad \text{(ii)}$$

is a tangent.

As the equation to the tangent at (x_1, y_1) of the circle (i) is

$$xx_1+yy_1-3(x+x_1)+2(y+y_1)-12=0,$$

or, $$(x_1-3)x+(y_1+2)y-(3x_1-2y_1+12)=0$$

this must be identical with (ii). Hence, comparing coefficients,

$$\frac{x_1-3}{4}=\frac{y_1+2}{3}=\frac{3x_1-2y_1+12}{31}$$

and each of these

$$=\frac{(3x_1-2y_1+12)-3(x_1-3)+2(y_1+2)}{31-3.4+2.3}=1$$

whence $x_1=7, y_1=1$.

Substituting these values in (i), we see that the equation is satisfied,

Hence, there is the definite point 7, 1 on the circle (i), at which the tangent is identical with the line (ii).

Hence, (ii) touches (i), and the point of contact is (7, 1).

Alternatively, the centre of the circle (i) is clearly, $(3, -2)$ and its radius is $\sqrt{(-3)^2+(2)^2-(-12)}=5$.

The straight line (ii) touches the circle (i) if the perpendicular distance from the centre on it is equal to the radius of the circle.

Now perpendicular from $(3, -2)$ on (ii) is

$$\frac{4.3+3.(-2)-31}{-\sqrt{4^2+3^2}}=5=\text{the radius of the circle.}$$

$\therefore$ (ii) touches (i).

Assuming the point of contact to be (x_1, y_1), comparing the equation of the tangent at the point with (ii), the co-ordinates x_1 and y_1 are obtained as before.

Ex. 4. *Prove that the locus of the middle points of any system of parallel chords of a circle is a diameter passing through the centre.*

Taking the centre of the circle as origin, the equation to the circle can be written as $x^2+y^2=a^2$ ... ... (i)

Let any one of the system of parallel chords be

$$y=mx+c \quad \cdots \quad \cdots \quad \text{(ii)}$$

where m is constant for all chords of the system, but c is different for different chord.

For points of intersection of (i) and (ii), eliminating y, the abscissæ are given by the equation

$$x^2+(mx+c)^2=a^2,$$

or, $$x^2(1+m^2)+2mcx+(c^2-a^2)=0.$$

Hence, if (x_1, y_1) and (x_2, y_2) be the extremities of the chord of intersection of (i) with (ii), x_1, x_2 being roots of the above equation,

$$x_1+x_2=-\frac{2mc}{1+m^2}.$$

Thus, X, Y being the middle point of the chord,

$$X=\tfrac{1}{2}(x_1+x_2)=-\frac{mc}{1+m^2}.$$

Also, X, Y being a point on (ii), $Y=mX+c$.

Eliminating c between these,

$$X=-\frac{m}{1+m^2}(Y-mX), \text{ or, } X+mY=0.$$

As this is free from c, this equation is satisfied by the co-ordinates of the middle point of every chord of the system, and it evidently represents the equation to a straight line passing through the origin *i.e.*, passing through the centre of the circle, and is thus a diameter.

Examples IV

1. Obtain the equation to a circle having its centre at (3, 7) and diameter 10.

What is the length of the intercept of this circle on the y-axis ? [*H. S. 1960, Compartmental*]

2. The extremities of a diameter of a circle have co-ordinates $(-4, 3)$ and $(12, -1)$. Find the equation to the circle. What length does it intercept on the y-axis ?

[*H. S. 1961, Compartmental*]

3. Show that the equation $3x^2 + 3y^2 - 5x - 6y + 4 = 0$ represents a circle, and find its radius and co-ordinates of its centre.

4. Obtain the equation to the circle passing through the points $(3, 4)$, $(3, -6)$, $(-1, 2)$, and determine its centre and radius. [*H. S. 1961*]

5. Obtain the co-ordinates of the centre of the circle passing through the points $(1, 2)$, $(3, -4)$, $(5, -6)$, and determine the length of its diameter.

Is the origin inside, or outside the circle ?

[*H. S. 1960*]

6. Find the equation to a circle which passes through the points $(0, -3)$ and $(3, -4)$, and which has its centre on the straight line $2x - 5y + 12 = 0$.

7. Find the equation to the circle passing through the origin and having intercepts 4 and -6 on the x-axis and y-axis respectively.

8. Find the equations to the circles which touch the axis of x and pass through the points $(1, -2)$ and $(3, -4)$.

9. A and B are two fixed points on a plane and the point R moves on the plane in such a way that $PA = 2PB$ always. Prove analytically that the locus of P is a circle.

[*H. S. 1961, Compartmental*]

10. B, C are fixed points having co-ordinates $(3, 0)$ and $(-3, 0)$ respectively. If the vertical angle BAC be $90°$, show that the locus of the centroid of the triangle ABC is a circle whose equation you are to determine.

[*H. S. 1961*]

11. (i). Find the length of the chord of the circle $x^2 + y^2 = 64$, intercepted on the straight line $3x + 4y - c = 0$.

(ii) Obtain the co-ordinates of the points of contact of any one of the two tangents to the above circle $x^2+y^2=64$, parallel to the line $3x+4y-c=0$. [*H. S. 1960*]

12. Prove that the straight line $y=x+a\sqrt{2}$ touches the circle $x^2+y^2=a^2$, and find its point of contact.

[*H. S. 1961*]

13. Show that the line $3x+4y+7=0$ touches the circle $x^2+y^2-4x-6y-12=0$, and find its point of contact.

14. Determine whether the straight line $x+y=2+\sqrt{2}$ touches the circle $x^2+y^2-2x-2y+1=0$. If it does, find the co-ordinates of the point of contact.

15. Find the equation to the circle

(i) having its centre at the point (3, 4), and touching the straight line $5x+12y+2=0$.

(ii) having its centre at $(1, -3)$ and touching the straight line $2x-y-4=0$.

16. Find the points at which the tangents to the circle $x^2+y^2-6x+8y=0$ is parallel to the line $3x+4y=0$.

17. Find the points on the circle $x^2+y^2-2x+6y-58=0$ at which the tangents are perpendicular to the line $4x-y=2$.

18. Show that the two circles

(i) $x^2+y^2+6x+14y+9=0$ and
$x^2+y^2-4x-10y-7=0$

touch each other externally.

(ii) $x^2+y^2-6x+6y-18=0$ and
$x^2+y^2-2y=0$

touch each other internally.

19. Find the length of the tangent drawn from

(i) the point $(-3, 11)$ to the circle
$$x^2+y^2-4x+2y-20=0.$$

(ii) the point (7, 2) to the circle

$$2x^2+2y^2+5x+y-15=0.$$

20. Show that the locus of the points from which the lengths of the tangents to the circles

$$x^2+y^2-3x+4y-7=0$$

and $$x^2+y^2+2x-5y+1=0$$

are equal, is a straight line perpendicular to the line joining the centres of the circles.

ANSWERS

1. $x^2+y^2-6x-14y+33=0$; 8.

2. $x^2+y^2-8x-2y-51=0$; $4\sqrt{13}$.

3. $\frac{1}{6}\sqrt{13}$; $(\frac{5}{6}, 1)$. **4.** $x^2+y^2-6x+2y-15=0$; $(3, -1)$; 5.

5. (11, 2); 20; outside. **6.** $x^2+y^2-8x-8y-33=0$.

7. $x^2+y^2-4x+6y=0$.

8. $x^2+y^2-6x+4y+9=0$, $x^2+y^2+10x+20y+25=0$.

10. $x^2+y^2=1$. **11.** (i) $\frac{2}{5}\sqrt{1600-c^2}$. (ii) $(\frac{24}{5}, \frac{32}{5})$ or $(-\frac{24}{5}, -\frac{32}{5})$.

12. $\left(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$. **13.** $(-1, -1)$. **14.** Yes; $\left(1+\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right)$.

15. (i) $x^2+y^2-6x-8y=0$. (ii) $5x^2+5y^2-10x+30y+49=0$.

16. (6, 0) and (0, −8). **17.** (3, 5) and (−1, −11).

19. (i) 12. (ii) 8.

CHAPTER V

CONICS

5·1. Definitions.

If a point moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the locus traced out by the point is defined to be a **conic.**

The fixed point is called the **focus** of the conic, and is usually denoted by the letter S.

The fixed straight line is referred to as the **directrix** of the conic.

The straight line through the focus perpendicular to the directrix is called the **axis.**

The constant ratio (of the distance of any point on a conic from the focus to the perpendicular distance of the point from the directrix) is called the **eccentricity** of the conic, and is usually denoted by the letter e.

When $e = 1$, the conic is defined to be a **parabola.**

When $e < 1$, the conic is defined to be an **ellipse.**

When $e > 1$, the conic is defined to be a **hyperbola.**

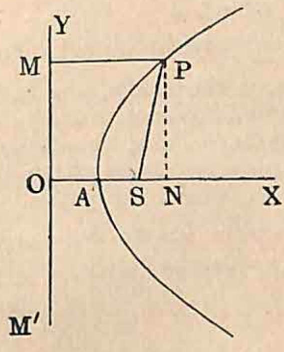
Note. The name 'Conic' or 'Conic-section' is due to the fact that these curves were first obtained and studied as sections of a cone by planes in various ways.

5·2. Parabola.

(A) *Equation with axis and directrix as axes of co-ordinates.*

Let S be the focus, and MM' the directrix of the parabola, which are fixed, and let OSX be the straight line

through S perpendicular to the directrix, so that it is the axis of the parabola, O being its point of intersection with the directrix.



Let us take OX as x-axis and OY (along the directrix) as y-axis, and let x, y be the co-ordinates of any point P on the parabola. If PN and PM be perpendiculars from P on OX and OY, then $PM = ON = x$, $PN = y$.

Let the distance OS of S from the directrix be d. Then co-ordinates of S are $(d, 0)$.

From the definition of the parabola,

$$\frac{PS}{PM} = 1, \quad \text{or,} \quad PS = PM. \quad \therefore \; PS^2 = PM^2,$$

$$\text{or,} \quad (x-d)^2 + y^2 = x^2. \quad \therefore \; y^2 = 2d(x - \tfrac{1}{2}d)$$

or writing $d = 2a$, this can be written as

$$\mathbf{y^2 = 4a\,(x - a)} \quad \ldots \quad \ldots \quad \text{(i)}$$

If A be the middle point of OS, clearly $OA = AS = a$.

The co-ordinates of A are then $(a, 0)$ and they evidently satisfy the equation (i). Thus, A is a point on the parabola. This point A is called the **vertex** of the parabola.

5

(B) *Standard form of the equation to a parabola.*

If we transfer the origin to the vertex A, the equation (i) of the parabola reduces to

$$y^2 = 4ax \qquad \ldots \qquad \ldots \text{ (ii)}$$

which is the standard form of the equation to a parabola.

Here the vertex is the origin, the axis of the parabola is the x-axis, and the line through the vertex parallel to the directrix is the y-axis, a being the distance of the vertex from the focus, which is also equal to the perpendicular distance of the vertex from the directrix.

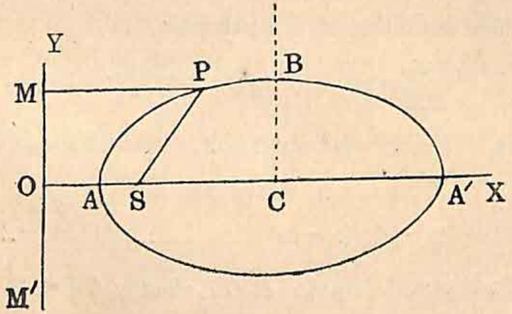
Note. For a discussion on the shape of the parabola and its elementary properties, see Chapter VI.

5·3. Ellipse.

(A) *Equation with directrix as y-axis, and perpendicular to it through the focus as x-axis.*

Let S be the focus, MM' the directrix, and $e\,(<1)$ be the eccentricity of the ellipse.

SO being perpendicular on MM', OSX is taken as the x-axis, and OY (along the directrix) as y-axis. Let (x, y) denote the co-ordinates of any point P on the ellipse. Let



the distance SO (of focus from the directrix) be d, and let PM be the perpendicular from P on the directrix, so that $PM = x$.

Now, from the definition of the ellipse,

$$\frac{PS}{PM}=e, \quad \text{or,} \quad PS=e.PM.$$

$$\therefore \quad PS^2=e^2.PM^2.$$

Hence, co-ordinates of S being evidently $(d, 0)$,

$$(x-d)^2+y^2=e^2x^2 \quad \ldots \quad \ldots \quad \text{(i)}$$

This is then the equation to the ellipse with directrix as y-axis, and perpendicular to it through the focus as x-axis, d being the distance of the focus from the directrix.

(B) *Standard form of the equation to an ellipse.*

The above equation (i) can be written in the form

$$x^2(1-e^2)-2dx+d^2+y^2=0,$$

$$\text{or,} \quad (1-e^2)\left(x-\frac{d}{1-e^2}\right)^2+y^2=\frac{d^2}{1-e^2}-d^2=\frac{d^2e^2}{1-e^2},$$

$$\text{or,} \quad \left(x-\frac{d}{1-e^2}\right)^2+\frac{y^2}{1-e^2}=\left(\frac{de}{1-e^2}\right)^2.$$

Writing $\dfrac{de}{1-e^2}=a$, and transferring the origin to C whose co-ordinates are $\left(\dfrac{d}{1-e^2}, 0\right)$ *i.e.*, $\left(\dfrac{a}{e}, 0\right)$, (the axes remaining parallel to their original directions), the equation to the ellipse reduces to its standard form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

$$\textit{i.e.,} \quad \mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \quad \ldots \quad \ldots \quad \text{(ii)}$$

where $b^2=a^2(1-e^2)$.

Here the origin C is on the axis perpendicular to the directrix through the focus, at a distance $\dfrac{d}{1-e^2}=\dfrac{a}{e}$ from the directrix, and the y-axis, CB is parallel to the directrix.

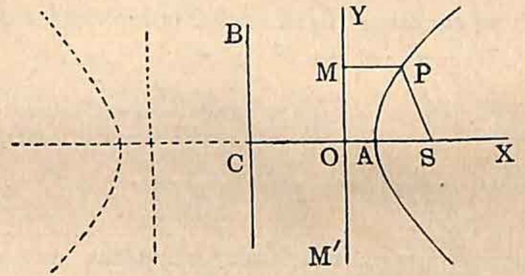
The point C is called the **centre** of the ellipse, the reason for which will be explained in Chapter VII.

Distance, $CS = \dfrac{d}{1-e^2} - d = \dfrac{e^2 d}{1-e^2} = ae.$

Note. For a discussion on the shape of the ellipse, and its elementary properties, see Chapter VII.

5·4. Hyperbola.

(A) *Equation with directrix as y-axis, and perpendicular to it through the focus as x-axis.*



Let S be the focus, MM' the directrix, and e (>1.) be the eccentricity of the hyperbola.

SO being perpendicular on MM', OSX is taken as the x-axis, and OY (along the directrix) as y-axis. Let (x, y) denote the co-ordinates of any point P on the hyperbola. Let the distance SO (of focus from the directrix) be d, and let PM be the perpendicular from P on the directrix, so that $PM = x$.

Now, from the definition of the hyperbola,

$$\frac{PS}{PM} = e, \quad \text{or,} \quad PS = e.PM.$$

$$\therefore \quad PS^2 = e^2.PM^2.$$

Hence, co-ordinates of S being evidently $(d, 0)$,

$$(x-d)^2 + y^2 = e^2 x^2 \quad \ldots \quad \ldots \quad \text{(i)}$$

This is then the equation to the hyperbola with directrix as y-axis, and perpendicular to it through the focus as x-axis, d being the distance of the focus from the directrix.

(B) *Standard form of the equation to a hyperbola.*

The above equation (i) can be written in the form (e being greater than 1 here),

$$x^2(e^2-1)+2dx-y^2=d^2\;;$$

$$\text{or,}\quad (e^2-1)\left(x+\frac{d}{e^2-1}\right)^2-y^2=d^2+\frac{d^2}{e^2-1}=\frac{e^2d^2}{e^2-1},$$

$$\text{or,}\quad \left(x+\frac{d}{e^2-1}\right)^2-\frac{y^2}{e^2-1}=\left(\frac{de}{e^2-1}\right)^2.$$

Writing $\dfrac{de}{e^2-1}=a$, and transferring the origin to C whose co-ordinates are $\left(-\dfrac{d}{e^2-1},\ 0\right)$ *i.e.*, $\left(-\dfrac{a}{e},\ 0\right)$, (the axes remaining parallel to their original directions) the equation to the hyperbola reduces to its standard form

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1,$$

$$\text{or,}\quad \mathbf{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

where $b^2=a^2(e^2-1)$.

Here the origin C is on the axis perpendicular through the focus to the directrix, at a distance $\dfrac{d}{e^2-1}=\dfrac{a}{e}$ from the directrix on the side opposite to the focus. This point C is called the *centre* of the hyperbola.

$$\text{Distance,}\quad CS=d+\frac{d}{e^2-1}=\frac{de^2}{e^2-1}=ae.$$

Note. For a discussion on the shape of the hyperbola, as also its elementary properties, see Chapter VIII.

5·5. Examples.

Ex. 1. *Find out the equation to the parabola whose focus is* $(-3, 4)$ *and directrix is* $6x-7y+5=0$. [*H. S. 1961*]

Let (x_1, y_1) be the co-ordinates of any point on the parabola.

Its distance from the given focus $(-3, 4)$ is $\sqrt{(x_1+3)^2+(y_1-4)^2}$, and its perpendicular distance from the given directrix $6x-7y+5=0$ is $\dfrac{6x_1-7y_1+5}{\sqrt{6^2+7^2}}$. For the parabola these two distances are equal.

Hence, $(x_1+3)^2+(y_1-4)^2=\dfrac{(6x_1-7y_1+5)^2}{6^2+7^2}$. Thus, the co-ordinates x_1, y_1 of any point on the parabola satisfy the equation

$$85\,\{(x+3)^2+(y-4)^2\}=(6x-7y+5)^2,$$

or, $\quad 49x^2+84xy+36y^2+450x-610y+2100=0$

which is then the required equation to the parabola.

Ex. 2. *Find the equation to the ellipse, whose focus is the point* $(-1, 1)$, *and directrix is the line* $x-y+3=0$, *and whose eccentricity is* $\frac{1}{2}$.

Let (x_1, y_1) be the co-ordinates of any point on the ellipse.

Its distance from the given focus $(-1, 1)$ is $\sqrt{(x_1+1)^2+(y_1-1)^2}$, and its perpendicular distance from the given directrix $x-y+3=0$ is $\dfrac{x_1-y_1+3}{\sqrt{1+1}}$. The ratio of these two distances is equal to the given eccentricity $\frac{1}{2}$ for any point on the ellipse.

Hence, $\quad \sqrt{(x_1+1)^2+(y_1-1)^2}=\dfrac{1}{2}\cdot\dfrac{x_1-y_1+3}{\sqrt{2}}$,

or, $\quad 8\{(x_1+1)^2+(y_1-1)^2\}=(x_1-y_1+3)^2$.

Thus, the co-ordinates x_1, y_1 of the any point on the ellipse satisfy the equation

$$8\,\{(x+1)^2+(y-1)^2\}=(x-y+3)^2,$$

or, $\quad 7x^2+2xy+7y^2+10x-10y+7=0$,

which is then the required equation of the ellipse in question.

CHAPTER VI

PARABOLA

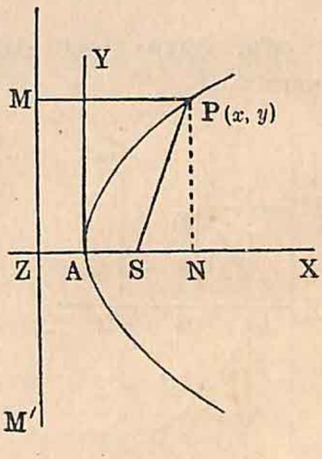
6·1. Parabola.

As has been defined in the previous chapter, a *parabola* is a curve traced out by a point which moves on a plane such that its distance from a fixed point on the plane is always equal to its perpendicular distance from a fixed straight line on that plane.

The fixed point is called the *focus*, and the fixed straight line is called the *directrix*.

6·2. Standard equation of a parabola.

Let S be the focus, and MM' the directrix of the parabola. Draw SZ perpendicular from S on MM', and let A be the middle point of ZS. Clearly, $\because$ $AS = AZ$, the point A is a point on the parabola. This point A is called the *vertex* of the parabola.



Let AS (the distance of the vertex from the focus) $= a$. Then AZ is also $= a$, and $SZ = 2a$.

Take A as origin, and ASX (perpendicular to the directrix through S) as x-axis, AY parallel to the directrix through A being the y-axis. Clearly the co-ordinates of S are $(a, 0)$.

P being any point on the parabola whose co-ordinates are (x, y), if PN be perpendicular on AX, and PM perpendicular to the directrix,

then $PM = ZN = AZ + AN = a + x$.

Now, from the definition of a parabola,

$$PS = PM, \quad \text{or,} \quad PS^2 = PM^2.$$

$$\therefore \quad (x-a)^2 + y^2 = (a+x)^2.$$

$$\therefore \quad \mathbf{y^2 = 4ax.} \quad \cdots \quad \cdots \quad \text{(i)}$$

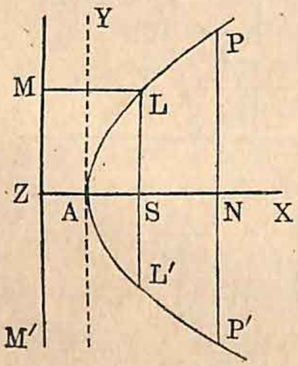
This being the relation satisfied by the co-ordinates of any point on the parabola, it represents the equation to the parabola in standard form with vertex as origin.

Here *a represents the distance of the vertex from the focus* (or from the directrix, the two being equal).

The line AX (perpendicular to the directrix through S), which is chosen as the axis of x here, is referred to as the *axis* of the parabola.

Note. In the previous chapter the equation was obtained with Z as origin, and then by transference of the origin to A, the equation to the parabola was obtained in the above standard form.

6·3. Shape and elementary properties of the parabola.



From the equation $y^2 = 4ax$, it is evident that if x be negative, y^2 being negative, y is imaginary. Hence, there is no real point with x negative, that is, there is no point of the parabola to the left of the origin A.

Again, for every positive value of x, there are two equal and opposite values of y. Hence corresponding to a point P on the parabola with positive ordinate PN, there is a point P' on the parabola with the same $x\ (= AN)$ with equal negative ordinate $P'N$. In other words, the parabola is

symmetrical with respect to the x-axis AX, which bisects every chord of the type PNP' perpendicular to it. As x diminishes, and ultimately becomes zero, the two ordinates, which are equal in value and opposite in sign, become zero, and the point coincides with the origin A, which is the vertex of the parabola. As x becomes larger and larger, the values of y also become larger in magnitude. Hence, the shape of the parabola is as shown in the figure, closed at the left end A, and open on the right, y gradually becoming numerically larger and larger with x, *the whole curve being symmetrical about* OX.

It is for this property that OX is defined as the axis, and A is called the vertex.

A chord PNP' perpendicular to the axis (*i.e.*, parallel to the directrix) and bisected by the axis, is called a *double ordinate.* PN or $P'N$ is the ordinate of P or P'.

The chord LSL' through the focus S parallel to the directrix (and so perpendicular to the axis) is called the **Latus Rectum.**

If LM be the perpendicular on the directrix from the extremity L of the latus rectum, from the property of the parabola, $LS=LM=ZS=2AS=2a$.

Thus, the **Latus rectum = 4a,**

i.e., the *latus rectum is four times the distance of the focus from the vertex*, or, double the distance of the focus from the directrix.

The equation $y^2=4ax$ asserts the geometrical property of the parabola, that $PN^2=4AS.AN$, or, *the square on the ordinate is equal to the rectangle contained by the abscissa and the latus rectum.*

To sum up, we note that for the standard equation $y^2=4ax$ of the parabola,

(i) the vertex is the origin ;

(ii) the length of the latus rectum is $4a$;

(iii) the focus has co-ordinates $(a, 0)$;

(iv) the directrix has equation $x = -a$;

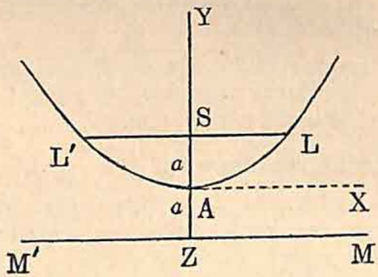
(v) the axis is the axis of x ;

and (vi) the co-ordinates of the extremities of the latus rectum are $(a, 2a)$ for L, and $(a, -2a)$ for L'.

Note. Equations $x^2=4ay$, $y^2=-4ax$, $x^2=-4ay$.

If in the equation $y^2=4ax$ of a parabola, the axes of x and y are interchanged, *i.e.*, choosing the vertex A as origin, and latus rectum being $4a$ as before, the axis of the parabola (the line perpendicular to the directrix through the focus) be taken along the y-axis, the x-axis being parallel to the directrix, the equation to the parabola becomes $x^2=4ay$, and the figure is as shown here.

Here, the co-ordinates of the focus are $(0, a)$ and equation to the directrix is $y=-a$. The co-ordinates of the extremities L and L' of the latus rectum are $(2a, a)$ and $(-2a, a)$ respectively.

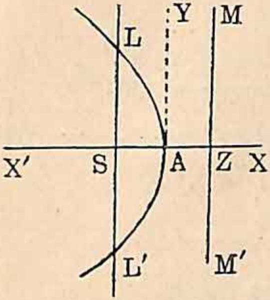


$x^2=4ay$

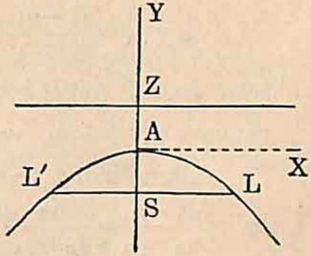
If in the equation $y^2=4ax$, the direction of the x-axis is reversed, *i.e.*, if the positive direction of the x-axis be taken from the vertex towards the directrix (the direction from the vertex to the focus being negative), the equation becomes $y^2=-4ax$, and the figure is as shown below, the concavity of the parabola being towards the negative side of the x-axis.

The co-ordinates of the focus are $(-a, 0)$ and directrix is $x=a$.

Similarly, in the equation $x^2=4ay$, if the direction of y-axis is reversed, the equation becomes $x^2=-4ay$, and the figure is shown



$y^2=-4ax$

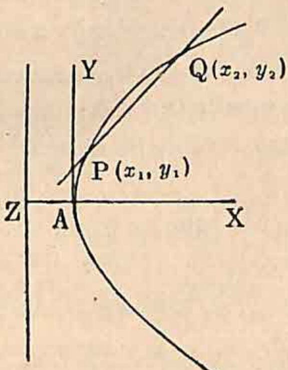


$x^2=-4ay$

above, the concavity of the parabola being towards the negative side of the y-axis.

The co-ordinates of the focus are $(0, -a)$ and directrix is $y=a$.

6·4. Equation to the tangent at a given point x_1, y_1 on the parabola $y^2=4ax$.



Let P be the point (x_1, y_1) on the parabola

$$y^2=4ax \qquad \cdots \qquad \cdots \quad \text{(i)}$$

and let $Q(x_2, y_2)$ be a neighbouring point on it.

The equation to the chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \cdots \quad \text{(ii)}$$

Now since P and Q both lie on the parabola (i),

we have $y_1^2 = 4ax_1 \quad \cdots \quad$ (iii) and $y_2^2 = 4ax_2 \quad \cdots \quad$ (iv)

$\therefore$ subtracting, $y_2^2 - y_1^2 = 4a(x_2 - x_1)$,

or, $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4a}{y_2 + y_1}$.

$\therefore$ equation (ii) can be written as

$$y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1) \qquad \cdots \quad \text{(v)}$$

Now make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position, the straight line PQ becomes the tangent at P, whose equation [*from (v)*] then becomes

$$y - y_1 = \frac{4a}{2y_1}(x - x_1), \quad \text{or,} \quad yy_1 - y_1^2 = 2a(x - x_1),$$

i.e., $yy_1 = y_1^2 + 2a(x - x_1) = 4ax_1 + 2a(x - x_1)$ [*by (iii)*].

Hence, the equation to the tangent at x_1, y_1 is

$$\mathbf{yy_1 = 2a(x + x_1).}$$

Cor. *The tangent at the vertex of the parabola* $y^2 = 4ax$ is the y-axis.

6·5. Equation to the normal at x_1, y_1 to the parabola $y^2 = 4ax$.

The tangent at x_1, y_1 to the parabola $y^2 = 4ax$ is

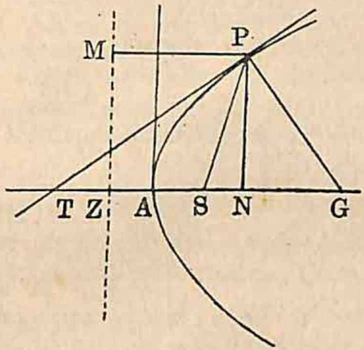
$$yy_1 = 2a(x + x_1), \text{ or, } y = \frac{2a}{y_1}(x + x_1),$$

of which the 'm' is $\dfrac{2a}{y_1}$.

The normal, which is perpendicular to the tangent, through x_1, y_1 is then

$$y - y_1 = -\frac{y_1}{2a}(x - x_1).$$

6·6. Tangent and normal properties: Subtangent and Subnormal.



The length of the axis intercepted between the tangent and the foot of the ordinate of any point on the parabola is defined as the *subtangent* of that point.

The length of the axis intercepted between the normal and the foot of the ordinate of any point on the parabola is defined as the *subnormal* of that point.

Thus, if PT and PG be the tangent and normal at P, intersecting the axis at T and G respectively, and PN be the ordinate of P, then

TN is the subtangent of P and NG is the subnormal.

Now from the equation $yy_1 = 2a(x + x_1)$ of the tangent at $P(x_1, y_1)$, the point T where it intersects the axis is obtained by putting $y = 0$, and thus for T, $x + x_1 = 0$ *i.e.*, $x = -x_1$.

Hence, $AT = AN$ in magnitude, T being on the negative side of A.

Hence, we get the geometrical property of the parabola that *the subtangent of any point on a parabola is bisected at the vertex.*

Again, in the equation $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ of the normal at P, putting $y = 0$, we get for the point G, $x - x_1 = 2a$, *i.e.*, $AG - AN = 2a$, or $NG = 2a =$ half the latus rectum. Hence, *the subnormal of any point of a parabola is constant and is equal to the semi-latus rectum.*

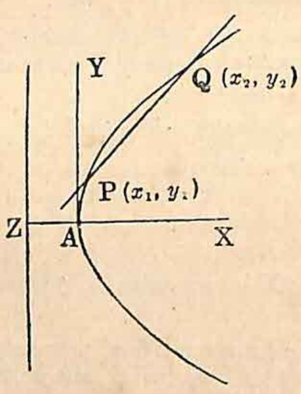
Further, $\because$ $AT = AN$ and $AS = AZ$, by adding we have $TS = ZN = PM$ (where PM is the perpendicular on the directrix) $= SP$. Hence, $\angle SPT = \angle PTS =$ the alternate $\angle TPM$. Also, $\because$ $\angle TPG$ is a right angle, $\angle SPG = \angle SGP$. Thus, we get further geometrical properties of the parabola, that :

(i) *the tangent at any point on a parabola bisects the angle between the focal distance of the point, and the perpendicular from the point to the directrix.*

(ii) *the tangent at any point on a parabola makes equal angles with the focal distance of the point and the axis.*

(iii) *the normal at any point on a parabola is equally inclined to the focal distance of the point and the axis.*

6·7. Length of the chord of the parabola $y^2 = 4ax$, intercepted by the straight line $y = mx + c$.



At the points of intersection of the line with the parabola, both the equations are satisfied. Hence, eliminating y between the two equations, the abscissæ of the points of intersection will be given by

$$(mx+c)^2 = 4ax,$$

$$\text{or,}\quad m^2x^2 + 2(mc-2a)x + c^2 = 0, \qquad \dots \quad \text{(i)}$$

which being a quadratic equation in x, there are only two values of x and accordingly only two points of intersection of the straight line with the parabola (which may be real and distinct, real and coincident, or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points P and Q of intersection. Then x_1 and x_2 are roots of (i).

$$\therefore \quad x_1 + x_2 = -\frac{2(mc-2a)}{m^2}, \text{ and } x_1x_2 = \frac{c^2}{m^2}.$$

$$\therefore \quad (x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2$$

$$= \frac{4(mc-2a)^2}{m^4} - \frac{4c^2}{m^2} = \frac{16(a^2-mca)}{m^4}.$$

Again, P and Q being on the given line,

$$y_1 = mx_1 + c, \quad y_2 = mx_2 + c.$$

$$\therefore \quad y_1 - y_2 = m(x_1 - x_2).$$

$\therefore$ length of the chord PQ

$$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = \sqrt{(x_1-x_2)^2(1+m^2)}$$

$$= \sqrt{\frac{16(a^2-mca)(1+m^2)}{m^4}} = \frac{\mathbf{4}}{\mathbf{m^2}}\sqrt{\mathbf{a(a-mc)(1+m^2)}}.$$

Cor. Condition of tangency.

The given line will touch the parabola only when the two points of intersection come into coincidence, *i.e.*, when the length of the chord intercepted is zero. Hence, the condition that the given line $y = mx + c$ may touch the parabola $y^2 = 4ax$ is

$$a - mc = 0, \text{ or, } \mathbf{c = \frac{a}{m}}.$$

6·8. *To show that $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all values of m, and to find the point of contact.*

The tangent at x_1, y_1 of the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$, or, $y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$ ... (i). If the line $y = mx + \frac{a}{m}$ ... (ii) be a tangent to the parabola at (x_1, y_1), the equations (i) and (ii) must be the same. Hence, comparing coefficients,

$$\frac{2a}{y_1} = m, \quad \frac{2ax_1}{y_1} = \frac{a}{m}. \qquad \therefore \quad x_1 = \frac{a}{m^2}, \quad y_1 = \frac{2a}{m}.$$

The line (ii) therefore will touch the parabola only if the assumed point x_1, y_1 is really a point on the parabola $y^2 = 4ax$, *i.e.*, if $\left(\frac{2a}{m}\right)^2 = 4a \cdot \frac{a}{m^2}$ which is evidently satisfied.

Thus, $y = mx + \frac{a}{m}$ touches the parabola, whatever m may be, and the point of contact is given by

$$x_1 = \frac{a}{m^2}, \quad y_1 = \frac{2a}{m}.$$

6·9. Co-ordinates of a point on the parabola $y^2 = 4ax$ expressed in terms of a single variable t.

We notice that if we substitute $x = at^2$, $y = 2at$ in the equation $y^2 = 4ax$ of the parabola, the equation is automatically satisfied for all values of t. Hence, any point on the parabola can have its two co-ordinates expressed in terms of a single variable t in the form

$$x = at^2, \quad y = 2at.$$

For different points, t will be different, and for a definite point on the parabola, t will be definite and unique. A point on the parabola will thus be referred to as the point t.

In working out many examples in parabola, when given by its standard form $y^2=4ax$, this assumption of the co-ordinates of a point on it in terms of the single variable t in the above form will be very helpful.

In this connection we may note that the equation to the tangent to the parabola $y^2=4ax$ at the point t is (See § 6·4)

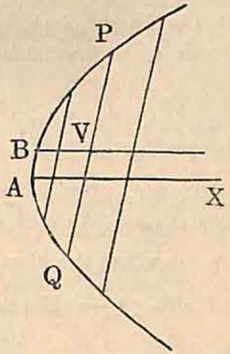
$$y.2at=2a(x+at^2),\ \text{or,}\ y=\frac{x}{t}+at.$$ Also the normal at the point t is (See § 6·5)

$$y-2at=-\frac{2at}{2a}(x-at^2),\ \text{or,}\ y+tx=2at+at^3.$$

Note. Significance of t.

From the equation of the tangent at t it is apparent that $\frac{1}{t}$ is the gradient of the tangent line at t, *i.e.*, t represents the cotangent of the angle made by the tangent line at t with the x-axis.

6·10. Locus of the middle points of a system of parallel chords ; diameter.



Let PQ, given by the equation $y = mx + c$ ... (i) be any one of a system of parallel chords of the parabola

$$y^2 = 4ax \quad \ldots \qquad \text{(ii)}.$$

As the chords are parallel, m is the same for all chords, but c is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating x, the ordinates are given by the roots of the equation

$$y^2 = 4a\left(\frac{y-c}{m}\right),$$

or, $$my^2 - 4ay + 4ac = 0.$$

If (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points of intersection P and Q, we get $y_1 + y_2 = \frac{4a}{m}$. Hence, for the middle point V,

$$\mathbf{y} = \tfrac{1}{2}(y_1 + y_2) = \frac{\mathbf{2a}}{\mathbf{m}}.$$

This being a relation free from c, it is satisfied by the middle point of every chord of the system.

Hence, this represents the locus of the middle points of the system of parallel chords, and we know that this represents a straight line parallel to the x-axis.

We thus see that *the locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to its axis.*

Such a straight line is defined to be a *diameter* of the parabola, bisecting the particular system of parallel chords.

For different m, *i.e.*, for differently directed systems of parallel chords we get different diameters.

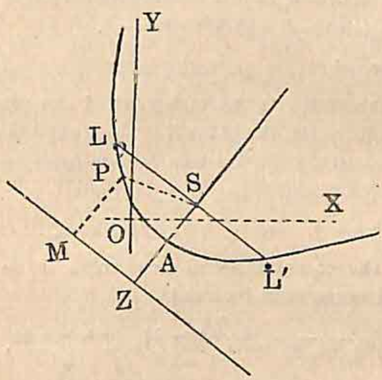
Note. If B be the point where the diameter in question meets the parabola, we have for B also, $y = \frac{2a}{m}$. Hence, $x = \frac{y^2}{4a} = \frac{a}{m^2}$. At this point then the tangent line is $y = mx + \frac{a}{m}$ [See § 6·8], and this is

also parallel to the system of chords bisected by the particular diameter. B is called the vertex of that diameter.

In fact any line parallel to the axis of the parabola is a diameter bisecting all chords parallel to the tangent at its extremity, *i.e.*, at the vertex of that diameter.

6·11. Illustrative Examples.

Ex. 1. *The focus of a parabola is* 6, 2 *and its vertex is* 3, −2. *Find the equation to the parabola, and the length of its latus rectum. Also obtain the co-ordinates of the extremities of its latus rectum.*



Let S be the focus having co-ordinates (6, 2), and A the vertex having co-ordinates (3, −2). The distance $AS=\sqrt{(6-3)^2+(2+2)^2}=5.$

Hence, the length of the latus rectum of the parabola $=4AS=20.$

Again, 'm' of the line joining A and S is $\frac{2-(-2)}{6-3}=\frac{4}{3}$, and this line AS is the axis of the parabola. The latus rectum is perpendicular to AS through S, and so its equation is

$$y-2=-\tfrac{3}{4}(x-6) \qquad \dots \qquad \dots \qquad \dots \text{ (i)}$$

If x_1, y_1 be the co-ordinates of an extremity (say L or L') of the latus rectum,

$\because$ $SL=$ semi-latus rectum $=10,$

$$(x_1-6)^2+(y_1-2)^2=100 \qquad \dots \qquad \dots \text{ (ii)}$$

and as (x_1, y_1) lies on (i),

$$y_1-2=-\tfrac{3}{4}(x_1-6) \qquad \dots \qquad \dots \qquad \dots \text{ (iii)}$$

From (ii) and (iii),

$$(x_1-6)^2(1+\tfrac{9}{16})=100.$$

$\therefore\ (x_1-6)^2=64.\quad \therefore\ x_1-6=\pm 8.$

Taking + sign, $x_1=14$, and from (iii), $y_1=-4$.

Taking − sign, $x_1=-2$, and from (iii), $y_1=8$.

Thus, the co-ordinates of the extremities of the latus rectum are $(14, -4)$ and $(-2, 8)$.

Lastly, produce SA to Z, such that $AZ=AS$. Then the co-ordinates of Z being (α, β) say, as A is the mid-point of ZS,

$$\tfrac{1}{2}(\alpha+6)=3 \text{ and } \tfrac{1}{2}(\beta+2)=-2.$$

$\therefore\ \alpha=0,\ \beta=-6.$

Now since the vertex is the mid-point of the perpendicular from the focus on the directrix, the point Z is clearly the foot of the perpendicular. The directrix of the parabola is therefore the line through Z perpendicular to ZAS, and hence its equation is

$$y+6=-\tfrac{3}{4}(x-0),\quad \text{or,}\quad 3x+4y+24=0. \quad \ldots \quad \ldots \quad \text{(iv)}$$

If x, y be the co-ordinates of any point P on the parabola, $\because\ PS=$ perp. distance from P on (iv)

$$\sqrt{(x-6)^2+(y-2)^2}=\frac{3x+4y+24}{\sqrt{3^2+4^2}}=\frac{3x+4y+24}{5},$$

whence $25\{(x-6)^2+(y-2)^2\}=(3x+4y+24)^2$
which is the equation to the parabola.

Ex. 2. *By suitably transferring the origin, show that the equation* $3y^2-10x-12y-18=0$ *reduces to the standard form of equation to a parabola, and hence obtain the co-ordinates of its vertex and focus, and the length of its latus rectum. Also determine the equation to its directrix.*

The given equation can be written as

$$3(y^2-4y)=10x+18,\quad \text{or,}\quad 3(y-2)^2=10(x+3).$$

Hence, transferring the origin to $(-3, 2)$, the equation reduces to the form $\quad y^2=\tfrac{10}{3}x \quad \ldots \quad \ldots \quad \ldots \quad$ (i)

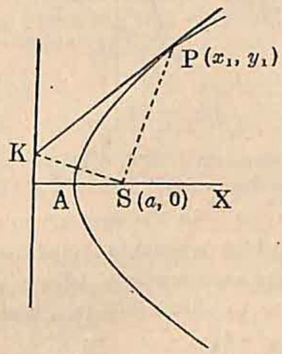
which is the standard form of the equation to a parabola. Comparing this with the equation $y^2=4ax$, whose latus rectum is $4a$, vertex is the

origin, focus is $(a, 0)$ and directrix is $x=-a$, we see that for the parabola (i), the latus rectum is $\frac{10}{3}$, the vertex is the new origin, and referred to this, the co-ordinates of the focus are $(\frac{5}{6}, 0)$, and the equation to the directrix is $x=-\frac{5}{6}$.

Hence, returning back to the given old origin, the co-ordinates of the vertex are $(-3, 2)$, the co-ordinates of the focus are $-3+\frac{5}{6}, 2+0$ *i.e.*, $(-2\frac{1}{6}, 2)$, and the equation to the directrix is $x=-\frac{5}{6}-3$ *i.e.*, $x=-3\frac{5}{6}$.

The latus rectum has already been shown to be $\frac{10}{3}$.

Ex. 3. *Prove that the length of any tangent to a parabola intercepted between its point of contact and the directrix subtends a right angle at the focus.*



Taking the vertex as origin and axis as x-axis, let the equation to the parabola be $y^2=4ax$. ... ... (i)

Then its focus S has co-ordinates $(a, 0)$ and the equation to the directrix is $x=-a$. ... ... (ii)

The tangent at any point $P(x_1, y_1)$ is given by

$$yy_1=2a(x+x_1) \quad \cdots \quad \cdots \text{(iii)}$$

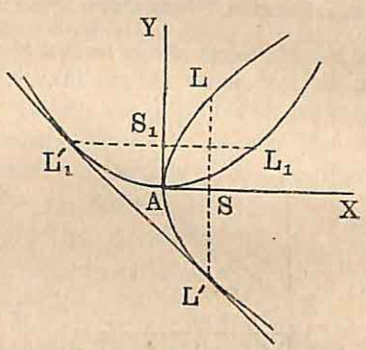
This meets the directrix (ii) at K, whose $x=-a$

$$\text{and} \quad y=\frac{2a}{y_1}(-a+x_1) \quad [\textit{from (iii)}].$$

Now the slope 'm' of the line $SP=\frac{y_1-0}{x_1-a}=\frac{y_1}{x_1-a}$ and the slope 'm'' of the line SK is $\frac{\frac{2a}{y_1}(x_1-a)-0}{-a-a}=-\frac{x_1-a}{y_1}$.

$\therefore\ mm'=-1$. Hence, SP and SK are at right angles, *i.e.*, PK subtends a right angle at S.

Ex. 4. *Two equal parabolas have the same vertex, and their axes are at right angles; prove that their common tangent touches each at an end of its latus rectum.*



Let $y^2=4ax$ ... (i) be the equation to one of the parabolas. The other parabola, which is equal to it (and hence has an equal latus rectum $4a$), having the same vertex A (chosen as origin) and having its axis perpendicular to that of (i) (*i.e.*, along the y-axis) is then given by

$$x^2=4ay \quad \cdots \quad \cdots \quad \cdots \text{(ii)}$$

Any tangent to (i) is $y=mx+\frac{a}{m}$ ... ... (iii)

having the point of contact at $\frac{a}{m^2}, 2am$.

If it be a tangent to (ii) also, the two points of intersection of (ii) and (iii) must coincide, or eliminating y, the two roots of

$$x^2-4a\left(mx+\frac{a}{m}\right)=0 \quad \cdots \quad \cdots \text{(iv)}$$

are equal, which requires

$$(4am)^2+4.\frac{4a^2}{m}=0,$$

$$\text{or,}\quad m^3=-1. \quad \therefore\ m=-1,$$

$\therefore$ the common tangent of (i) and (ii) is $y=-x-a$,

or, $x+y+a=0$.

The point of contact of this common tangent on (i) is (putting $m=-1$ here) $a, -2a$, which is clearly the co-ordinates of one extremity L' of the latus rectum.

The point of contact of the common tangent on (ii) [$\because$ for the equal roots for (iv) the sum of the roots is $4am=-4a$] is given by $x=-2a$, and hence from (ii) $y=a$. But $-2a, a$ are clearly the co-ordinates of the extremity L'_1, of the latus rectum of (ii).

Hence, the common tangent of the two parabolas touch each at an end of its latus rectum.

Examples VI

1. Find the point on the parabola $y^2=18x$ at which the ordinate is three times the abscissa.

2. The parabola $y^2=4ax$ passes through the point $(2, -6)$. Find the length of its latus rectum.

3. Find the equation to the line joining the vertex to the positive end of the latus rectum of the parabola $y^2=8x$.

4. A double ordinate of the parabola $y^2=4ax$ is of length $8a$. Prove that the line joining the vertex to its two ends are at right angles. [*H. S. 1960*]

5. Find the latus rectum of the parabola whose focus is $(2, -3)$, and directrix is $5x-12y+6=0$.

6. Find the equation to the parabola

(i) whose focus is $(5, 3)$ and directrix is $3x-4y+1=0$.

(ii) whose focus is $(-6, -6)$ and vertex is $(-2, 2)$.

7. Find the vertex, focus and latus rectum of each of the parabolas

(i) $y^2=4(x+y)$. (ii) $x^2+2y=8x-7$.

8. Find out the equation of the tangent to the parabola $y^2=4ax$ at the extremity of the latus rectum. [*H. S. 1960*]

9. Find the equation to the tangent to the parabola

(i) $y^2 = 9x$ at the point whose ordinate is 6.

(ii) $y^2 = 12x$ at the positive extremity of the latus rectum.

10. Show that the foot of the perpendicular from the focus of the parabola $y^2 = 4ax$ on any tangent lies on the y-axis. [*H. S. 1961, Compartmental*]

11. Prove that the tangents at the extremities of the latus rectum of a parabola meet on the directrix, and are at right angles.

12. The two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles. Find the locus of P.

13. (i) Prove that any two perpendicular tangents to the parabola $y^2 = 4ax$ intersect on the directrix.

(ii) If two tangents to a parabola are at right angles, show that their points of contact are at the extremities of a focal chord.

14. A tangent to the parabola $y^2 = 12x$ makes an angle $45°$ with the axis. Find the co-ordinates of its point of contact.

15. A tangent to the parabola $y^2 = 4ax$ makes an angle $60°$ with the axis. Find its point of contact.

16. Find the equation to the tangent to the parabola $y^2 = 7x$ which is parallel to the straight line $x - 4y - 3 = 0$. Find also its point of contact.

17. Find the equation of the tangent to the parabola $y^2 = 8x$ which is perpendicular to $x + 2y + 7 = 0$.

18. Find the point on the parabola $y^2 = 8x$ at which the normal is inclined at an argle $60°$ with the positive direction of the x-axis.

19. Find the equation to the locus of the foot of the perpendicular from the vertex on the tangent at any point of the parabola $y^2 = 4ax$.

20. Find the equation to the chord of the parabola $y^2 = 8x$ which is bisected at the point $(2, -3)$.

21. Prove that the locus of the middle points of all

chords of the parabola $y^2 = 4ax$ which are drawn through the vertex is the parabola $y^2 = 2ax$.

22. Find the length of the chord of the parabola $y^2 = 12x$ which is inclined at an angle of $45°$ with the axis, and passes through the point (1, 3).

23. Find the length of the chord of the parabola $y^2 = 20x$ along the straight line $x - 2y + 4 = 0$.

24. Find the length of the normal chord of the parabola $y^2 = 4ax$ through an extremity of the latus rectum.

25. Find the middle point of the line $3y - 4x = 4$ intercepted by the parabola $y^2 = 8x$.

26. Prove that the product of the ordinates of the extremities of a focal chord of a parabola is constant, and deduce that the normals at the extremities of any focal chord are at right angles.

27. Prove that the normal chord of a parabola at the point whose ordinate is equal to its abscissa subtends a right angle at the focus.

28. Find the equation to the common tangent of the parabolas $y^2 = 32x$ and $x^2 = 4y$.

29. Prove that the common tangents of the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 - 2ax = 3a^2$ are both inclined at $30°$ to the x-axis.

30. Show that the sum of the ordinates of the extremities of any chord of a parallel system is constant.

ANSWERS

1. (2, 6). **2.** 18. **3.** $y = 2x$. **5.** 8.

6. (i) $25\{(x-5)^2 + (y-3)^2\} = (3x - 4y + 1)^2$.

(ii) $4x^2 - 4xy + y^2 + 104x + 148y - 124 = 0$.

7. (i) $(-1, 2)$; $(0, 2)$; 4. (ii) $(4, 4\frac{1}{2})$; $(4, 4)$; 2.

8. $y = \pm(x + a)$. **9.** (i) $3x - 4y + 12 = 0$. (ii) $y = x + 3$.

12. $x = -1$. **14.** (3, 6). **15.** $\left(\frac{a}{3}, \frac{2a}{\sqrt{3}}\right)$.

16. $x - 4y + 28 = 0$; (28, 14). **17.** $y = 2x + 1$.

18. $(6, -4\sqrt{3})$. **19.** $x(x^2 + y^2) + ay^2 = 0$.

20. $4x + 3y + 1 = 0$. **22.** $4\sqrt{6}$. **23.** 80.

24. $8a\sqrt{2}$. **25.** $(\frac{5}{4}, 3)$. **28.** $2x + y + 4 = 0$.

CHAPTER VII

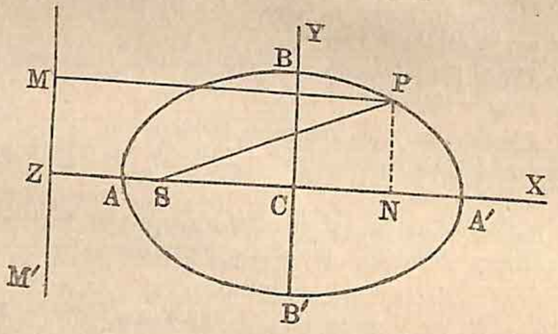
ELLIPSE

7·1. Ellipse.

An *ellipse* is a curve traced out by a point which moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the ratio being less than unity.

The fixed point is called the *focus*, the fixed straight line is called the *directrix*, and the constant ratio (less than unity in this case) is called the *eccentricity* of the ellipse.

7·2. Standard equation of an ellipse.



Let S be the focus, MM' the directrix, and $e(<1)$ the given eccentricity of the ellipse.

Draw SZ perpendicular from S on MM', and let it be divided internally at A and externally at A' in the ratio $e:1$. As $e<1$, $SA'<A'Z$, and accordingly A' is to the right of S as in the figure, on the same side of the directrix MZM' as A, S being between A and A'. Then $SA=e.AZ$ and $SA'=e.A'Z$. Hence, by definition of the ellipse, A and A' are points on the ellipse.

Let C be the middle point of AA'.

Thus, $SA+SA'=e(AZ+A'Z)$ and $SA'-SA=e(A'Z-AZ)$.

Hence, AA' or $2CA=e.\,2CZ$ and $2CS=e.AA'=e.2CA$.

Let $\quad CA(=CA')=a$.

Then $CZ=\dfrac{a}{e}$ and $CS=ae$. Let us choose C as origin, and CX along AA' as x-axis, the y-axis CY being $B'CB$ perpendicular to AA' through C.

Now P being any point on the ellipse whose co-ordinates are (x, y), let PN be the perpendicular from P to the x-axis AA', and PM be perpendicular to the directrix MM'. Then $CN=x$, $PM=ZN=ZC+CN=\dfrac{a}{e}+x$. Also co-ordinates of S are evidently $-ae$, 0 ($\because CS=ae$).

Hence, from the property of the ellipse,

$$SP=e.\,PM \quad \text{or} \quad SP^2=e^2.\,PM^2.$$

$$\therefore \quad (x+ae)^2+y^2=e^2\left(\frac{a}{e}+x\right)^2,$$

or, $\quad x^2(1-e^2)+y^2=a^2(1-e^2)$, ($\because e<1$ here)

or, $\quad$ writing $a^2(1-e^2)=b^2$,

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}=\mathbf{1} \qquad \cdots \qquad \cdots \quad \text{(i)}$$

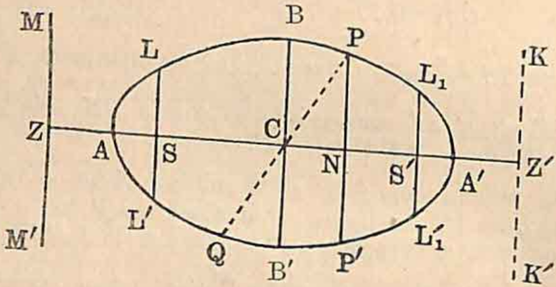
This being the relation satisfied by the co-ordinates of any point on the ellipse, it represents the equation to the ellipse in its standard form.

Here, C, the middle point of AA' (called the *centre*) is the origin, $CA=CA'=\frac{1}{2}AA'=a$, and $b^2=a^2(1-e^2)$.

7·3. Shape and elementary properties of the ellipse.

From the equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, it is apparent that corresponding to any value of x, there are two equal and

opposite values of y, namely $\pm \frac{b}{a}\sqrt{a^2-x^2}$. Hence, on a line perpendicular to AA', corresponding to any point P on one side of it, there is another symmetrical point P' on the



other side. Thus every line perpendicular to AA' is bisected by it, and accordingly *the curve is symmetrical with respect to the x-axis.*

Similarly, for every value of y we get two equal and opposite values of x, and thus *the curve is symmetrical with respect to the y-axis also.* Accordingly if we take points S', Z' on the x-axis, such that $CS'=CS$ and $CZ'=CZ$ on opposite sides of C, and draw $KZ'K'$ parallel to MZM', the ellipse, from symmetry about BCB', can as well be described with S' as focus and KK' as directrix, e being the same. Thus *there is a second focus and a second directrix of the ellipse* symmetrically situated with respect to C.

Again, from the equation to the ellipse, for $y=0$ we get $x=\pm a$. Hence, the ellipse cuts the x-axis at points A' and A given by $x=a$ and $x=-a$ respectively. Similarly, for $x=0$, we get $y=\pm b$, so that the ellipse cuts the y-axis at B and B' given by $y=b$ at B and $y=-b$ at B', so that $CB=CB'=b$ in length.

Moreover, from the equation of the ellipse, if $x>a$ or $<-a$, $\frac{x^2}{a^2}>1$ and so y^2 is negative, and hence y is imagi-

nary. Thus there are no points of the ellipse beyond A' to the right, or beyond A to the left. Similarly, if $y > b$ or $< -b$, x is imaginary, and thus there are no points of the ellipse above B or below B' in the y-direction. Hence, *the ellipse is limited in all directions, and is a closed curve.*

Lastly, from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of the ellipse, if x_1, y_1 be the co-ordinates of a point P on the ellipse which satisfy the equation, the co-ordinates $-x_1, -y_1$ will also satisfy it, and accordingly the diagonally opposite point Q, where PQ is bisected at C, is also a point on the ellipse. Thus *every chord of the ellipse through C is bisected at C*, and thus the ellipse is symmetrical with respect to the origin C, the mid-point of AA' or BB'. That is why C is called the *centre* of the ellipse.

The length $AA' = 2a$ along the x-axis is called the *major axis* of the ellipse.

The length $BB' = 2b$ along the y-axis is called the *minor axis* of the ellipse.

The points A, A', B, B' are called the *vertices* of the ellipse.

The chord LSL' through the focus S, or $L_1S'L_1'$ through the focut S', perpendicular to the major axis AA' (*i.e.*, parallel to the directrix) is called the *latus rectum* of the ellipse, both being of same length by symmetry.

Now ae being the length CS', the x-co-ordinate of the extremity L_1 or L'_1 of the latus rectum is ae. Hence, from the equation to the eilipse, the y-co-ordinate of L_1 or L'_1 is given by $\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$.

Hence, $y = \pm b\sqrt{1-e^2} = \pm a(1-e^2)$.

Thus the *length* $L_1L'_1$, or LL' of the latus rectum

$$= 2a(1-e^2) = 2\frac{b^2}{a}.$$

$\therefore$ Semi-latus rectum $= \dfrac{b^2}{a} = a\,(1-e^2)$.

Co-ordinates of the extremity L_1, of the latus rectum are ae, $a\,(1-e^2)$.

The eccentricity of the ellipse is given by

$$\mathbf{b^2 = a^2\,(1-e^2)} \text{ or } \mathbf{e^2 = \frac{a^2-b^2}{a^2}}.$$

Length of the focal distances S′P, SP of any point P on the ellipse :

Let the co-ordinates of P be (x_1, y_1). Those of S' being $(ae, 0)$, we get

$$S'P^2 = (x_1 - ae)^2 + y_1{}^2 = (x_1 - ae)^2 + b^2\left(1 - \frac{x_1{}^2}{a^2}\right)$$

[from the equation to the ellipse]

$$= (x_1 - ae)^2 + (1-e^2)(a^2 - x_1{}^2)$$

[$\because \; b^2 = a^2\,(1-e^2)$]

$$= e^2x_1{}^2 - 2x_1ae + a^2 = (a - ex_1)^2.$$

$\therefore$ $\mathbf{S'P = a - ex_1}$, which is the positive value of $S'P$, $\because$ $x_1 < a$, as also $e < 1$.

Similarly, $\mathbf{SP = a + ex_1}$.

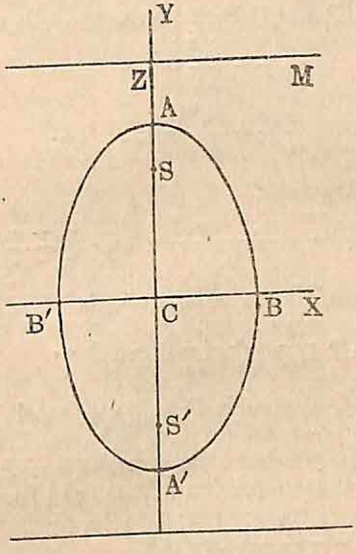
Thus, $SP + S'P = 2a =$ length of the major axis. Hence, we get an important property of the ellipse, namely, *the sum of the focal distances of any point on the ellipse is constant and equal to the major axis.*

Cor. Focal distance of an extremity of the minor axis is equal to the semi-major axis.

Note 1. As the whole figure is symmetrical with respect to the minor axis BCB', henceforth, for convenience, as a matter of convention, we shall denote the right-hand focus $(ae, 0)$ as S, the right-hand vertex $(a, 0)$ as A, and the left-hand focus $(-ae, 0)$ as S', the left-hand vertex $(-a, 0)$ as A', the right-hand directrix having equation $x = \dfrac{a}{e}$ being denoted by MZM', and left-hand directrix $x = -\dfrac{a}{e}$ as $KZ'K'$.

Note 2. The equation $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, $a > b$.

If in the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the axes of x and y be interchanged, the equations becomes $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ or $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. Here, a being greater than b, the major axis having length $2a$ is along the y-axis, and the



minor axis of length $2b$ is along the x-axis. The foci, being on the major axis, *i.e.*, y-axis, will have co-ordinates, $0, \pm\sqrt{a^2-b^2}$. The eccentricity is as before $e=\sqrt{(a^2-b^2)}/a$. The directrices being parallel to the minor axis *i.e.*, x-axis here, are given by $y=\pm\frac{a}{e}$.

7·4. Equation to the tangent at a given point x_1, y_1, on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let P be the point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \cdots \quad \cdots \quad (1).$$

and let $Q\ (x_2, y_2)$ be a neighbouring point on it. The equation to the chord PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \cdots \quad \text{(ii)}$$

Now since P and Q both lie on the ellipse (i), we have,

$$\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}=1 \quad \cdots \quad \cdots \quad \text{(iii)}$$

$$\frac{x_2^{\ 2}}{a^2}+\frac{y_2^{\ 2}}{b^2}=1. \quad \cdots \quad \cdots \quad \text{(iv)}$$

Hence, subtracting,

$$\frac{x_2^{\ 2}-x_1^{\ 2}}{a^2}+\frac{y_2^{\ 2}-y_1^{\ 2}}{b^2}=0, \text{ or } \frac{y_2-y_1}{x_2-x_1}=-\frac{b^2}{a^2}.\frac{x_2+x_1}{y_2+y_1}.$$

$\therefore$ equation (ii) can be written as

$$y-y_1=-\frac{b^2}{a^2}.\frac{x_2+x_1}{y_2+y_1}(x-x_1) \cdots \quad \cdots \quad \text{(v)}$$

Now make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position, the straight line PQ becomes the tangent at P, whose equation [*from (v)*] then becomes

$$y-y_1=-\frac{b^2}{a^2}.\frac{x_1}{y_1}(x-x_1),$$

$$\text{or, } (y-y_1)\frac{y_1}{b^2}+\frac{x_1}{a^2}(x-x_1)=0,$$

$$\text{or, } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}=1. \quad [\ by\ (iii)\ ]$$

Hence, the equation to the tangent at x_1, y_1 to the ellipse (i) is

$$\mathbf{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.}$$

7·5. Equation to the normal at x_1, y_1 to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

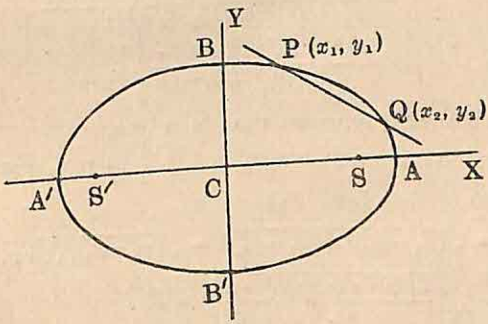
The tangent at (x_1, y_1) to the ellipse is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$, or $y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}$, of which the '$m$' is $-\frac{b^2x_1}{a^2y_1}$.

The normal, which is perpendicular to the tangent through x_1, y_1, has its 'm' $=\frac{a^2y_1}{b^2x_1}$, and accordingly its equation is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1),$$

or $$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}.$$

7·6. Length of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, intercepted by the straight line $y=mx+c$.



At the points of intersection of the line with the ellipse, both the equations are satisfied. Hence, eliminating

y between the two equations, the abscissæ of the points of intersection will be given by

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

$$\text{or,}\quad x^2\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)+\frac{2mc}{b^2}x+\left(\frac{c^2}{b^2}-1\right)=0, \qquad \cdots \quad \text{(i)}$$

which being a quadratic equation in x, there are only two values of x and accordingly, only two points of intersection of the given straight line with the ellipse, (real, coincident, or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points P and Q of intersection. Then x_1 and x_2 are the roots of (i).

$$\therefore \quad x_1+x_2=-\frac{2mc}{b^2}\Big/\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)=-\frac{2mca^2}{a^2m^2+b^2}.$$

$$\text{and}\quad x_1x_2=\left(\frac{c^2}{b^2}-1\right)\Big/\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)=\frac{a^2(c^2-b^2)}{a^2m^2+b^2}.$$

$$\begin{aligned}\therefore\quad (x_1-x_2)^2&=(x_1+x_2)^2-4x_1x_2\\
&=\frac{4m^2c^2a^4}{(a^2m^2+b^2)^2}-\frac{4a^2(c^2-b^2)}{a^2m^2+b^2}\\
&=\frac{4a^2\{m^2c^2a^2-(c^2-b^2)(a^2m^2+b^2)\}}{(a^2m^2+b^2)^2}\\
&=\frac{4a^2b^2(a^2m^2+b^2-c^2)}{(a^2m^2+b^2)^2}.\end{aligned}$$

Again P and Q lying on the given line,

$$y_1=mx_1+c,\ y_2=mx_2+c. \quad \therefore\ y_1-y_2=m(x_1-x_2).$$

$\therefore$ length of the chord PQ

$$\begin{aligned}&=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(x_1-x_2)^2(1+m^2)}\\
&=\sqrt{\frac{4a^2b^2(a^2m^2+b^2-c^2)(1+m^2)}{(a^2m^2+b^2)^2}}\\
&=\mathbf{\frac{2ab\sqrt{1+m^2}\sqrt{a^2m^2+b^2-c^2}}{a^2m^2+b^2}}.\end{aligned}$$

Cor. Condition of tangency.

The given line will touch the ellipse only when the two points of intersection come into coincidence, *i.e.*, when the length of the chord intercepted is zero. Hence, the condition that the given line $y=mx+c$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $a^2m^2+b^2-c^2=0$, or $\mathbf{c=\pm\sqrt{a^2m^2+b^2}}$.

7·7. *To show that $y=mx+\sqrt{a^2m^2+b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m, and to find the point of contact.*

The tangent at x_1, y_1 of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\text{is } \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \text{ or, } y=-\frac{b^2x_1}{a^2y_1}x+\frac{b^2}{y_1}. \quad \cdots \quad \text{(i)}$$

If the line $y=mx+\sqrt{a^2m^2+b^2}$ ... (ii) be a tangent to the ellipse at (x_1, y_1), the equations (i) and (ii) must be same. Hence, comparing coefficients,

$$-\frac{b^2x_1}{a^2y_1}=m, \quad \frac{b^2}{y_1}=\sqrt{a^2m^2+b^2}.$$

$$\therefore \quad y_1=\frac{b^2}{\sqrt{a^2m^2+b^2}}, \quad x_1=-\frac{a^2my_1}{b^2}=-\frac{a^2m}{\sqrt{a^2m^2+b^2}}.$$

The line (ii) therefore will touch the ellipse only if the assumed point x_1, y_1 is really a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

$$i.e., \text{ if } \left(-\frac{am}{\sqrt{a^2m^2+b^2}}\right)^2+\left(\frac{b}{\sqrt{a^2m^2+b^2}}\right)^2=1$$

which is evidently satisfied.

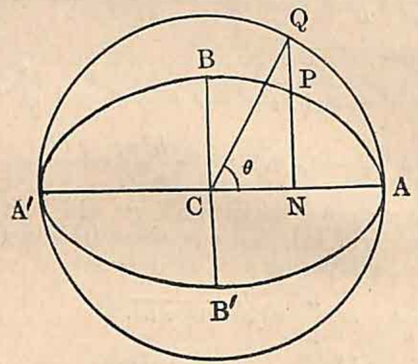
Thus, $y=mx+\sqrt{a^2m^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, whatever m may be, and the point of contact is given by

$$\mathbf{x_1=-\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \quad y_1=\frac{b^2}{\sqrt{a^2m^2+b^2}}}.$$

Similarly, $y = mx - \sqrt{a^2m^2+b^2}$ is also a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, for all values of m, the co-ordinates of the point of contact being

$$\frac{a^2m}{\sqrt{a^2m^2+b^2}}, -\frac{b^2}{\sqrt{a^2m^2+b^2}}.$$

7·8. Auxiliary circle.



The circle on the major axis AA' of an ellipse as diameter is defined as *the auxiliary circle* of the ellipse.

The centre of the circle being the origin C, and its radius being the semi-major axis a, the *equation to the auxiliary circle is*

$$\mathbf{x^2+y^2=a^2}.$$

Let PN be an ordinate of the ellipse, which, when produced, meets the auxiliary circle at Q.

Then x being the abscissa CN of the point P, from the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of the ellipse, the ordinate of the ellipse

$$PN = y = \sqrt{b^2\left(1-\frac{x^2}{a^2}\right)} = \frac{b}{a}\sqrt{a^2-x^2}.$$

Also, from the equation $x^2+y^2=a^2$ of the auxiliary circle, the abscissa $CN=x$ being the same, the ordinate $QN=\sqrt{a^2-x^2}$.

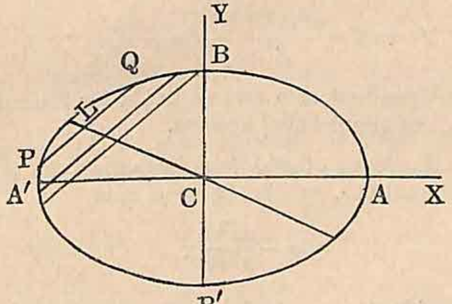
Thus, $\dfrac{PN}{QN}=\dfrac{b}{a}$.

Hence, the *ratio of any ordinate of the ellipse to the corresponding ordinate of its auxiliary circle is always the same*, and is equal to the ratio of the minor axis to the major axis of the ellipse.

Note. *Co-ordinates of any point on an ellipse expressed in terms of a single variable*; **eccentric angle** *of a point on the ellipse.*

Let $\angle QCN=\theta$. Then, $\because$ $CQ=a$, clearly $CN=a\cos\theta$, $NQ=a\sin\theta$. $\therefore$ $NP=\dfrac{b}{a}\cdot a\sin\theta=b\sin\theta$. Thus, co-ordinates of any point P on the ellipse can be written as $\mathbf{a\cos\theta}$, $\mathbf{b\sin\theta}$, in terms of a single variable θ. Here, θ is called the *eccentric angle* of the point P on the ellipse.

7·9. Locus of the middle points of a system of parallel chords; diameter.



Let PQ, given by the equation

$$y=mx+c \qquad \cdots \quad \text{(i)}$$

be anyone of a system of parallel chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \cdots \quad \text{(ii)}$$

As the chords are parallel, m is the same for all chords, but c is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating y, the abscissæ are given by the roots of the equation

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

or, $(a^2m^2+b^2)x^2+2a^2mcx+a^2(c^2-b^2)=0. \qquad \cdots \quad$ (iii)

Thus, if (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q, x_1, x_2 are the roots of (iii), and so $x_1+x_2=-\dfrac{2a^2mc}{a^2m^2+b^2}$.

Hence, if X, Y be the co-ordinates of the mid-point L of PQ,

$$X=\tfrac{1}{2}(x_1+x_2)=-\frac{a^2mc}{a^2m^2+b^2}.$$

Also, $\because$ L is a point on (i), $Y=mX+c$.

$\therefore$ eliminating c,

$$Y=mX-\frac{a^2m^2+b^2}{a^2m}X=-\frac{b^2}{a^2m}X,$$

which is independent of c, and so holds for the middle point of any chord of the parallel system.

Hence, the locus of middle points of a system of parallel chords of the ellipse, parallel to $y=mx$, is

$$\mathbf{y=-\frac{b^2}{a^2m}x}$$

which is evidently a straight line passing through the origin, *i.e.*, the centre C of the ellipse. This straight line is called a *diameter* of the ellipse. For different values of m (*i.e.*, for differently directed system of parallel chords) we get different diameters, all passing through the centre.

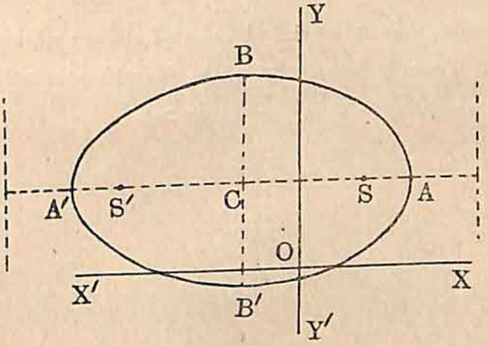
7·10. Illustrative Examples.

Ex.1. *Show that the equation* $5x^2+9y^2+10x-36y-4=0$ *represents an ellipse, and find its eccentricity, latus rectum, and co-ordinates of the foci. Find also the equations to its directrices.*

The given equation can be written as

$$5(x^2+2x)+9(y^2-4y)=4, \quad \text{or,} \quad 5(x+1)^2+9(y-2)^2=45,$$

i.e.,
$$\frac{(x+1)^2}{9}+\frac{(y-2)^2}{5}=1.$$



Transferring the origin to the point $-1, 2$, the equation reduces to

$$\frac{x^2}{9}+\frac{y^2}{5}=1 \quad \cdots \quad \cdots \quad \text{(i)}$$

which is the standard form of the equation to an ellipse, with centre as origin.

Hence, the given equation represents an ellipse whose centre (the new origin) has co-ordinates $-1, 2$ referred to the original axes of co-ordinates.

Comparing equation (i) with the standard equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ we notice that for (i), } a^2=9, b^2=5.$$

Thus, eccentricity of the given ellipse is

$$e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{9-5}{9}}=\frac{2}{3}.$$

Latus rectum $=2\dfrac{b^2}{a}=2\cdot\dfrac{5}{3}=3\tfrac{1}{3}$.

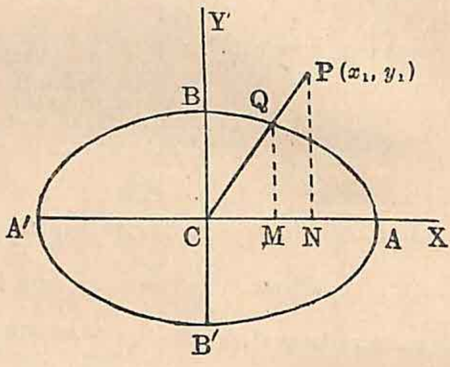
Co-ordinates of the foci referred to the centre are

$$\pm ae, 0, \text{ i.e., } \pm 3.\tfrac{2}{3}, 0, \text{ or } \pm 2, 0.$$

Hence, referred to the original axes, the co-ordinates of the foci are $(-1+2, 2+0)$ and $(-1-2, 2+0)$ *i.e.*, $(1, 2)$ and $(-3, 2)$ respectively.

Also, the equations to the directrices referred to centre are $x=\pm\dfrac{a}{e}$ or $x=\pm\dfrac{3}{\frac{2}{3}}=\pm\dfrac{9}{2}$. Hence, referred to original axes, the equation to the directrices are $x=\pm\frac{9}{2}-1$ *i.e.*, $x=\frac{7}{2}$ and $x=-\frac{11}{2}$ respectively.

Ex. 2. *Prove that the point* x_1, y_1 *is inside or outside the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *according as* $\dfrac{x_1^{\,2}}{a^2}+\dfrac{y_1^{\,2}}{b^2}<1$ *or* >1.



Let P be the point where co-ordinates are x_1, y_1 and let CP, the line joining the centre (origin) to P intersect the ellipse at Q. Then, if $\dfrac{CP}{CQ}=\lambda$, clearly P is outside the ellipse if $\lambda>1$ and inside if $\lambda<1$.

Now, PN and QM being perpendiculars on the x-axis CAX, clearly $x_1=CN$, $y_1=NP$, and $\dfrac{CM}{CN}=\dfrac{MQ}{NP}=\dfrac{CQ}{CP}=\dfrac{1}{\lambda}$.

$\therefore$ The co-ordinates of Q, namely CM and MQ are respectively $\dfrac{x_1}{\lambda}, \dfrac{y_1}{\lambda}$.

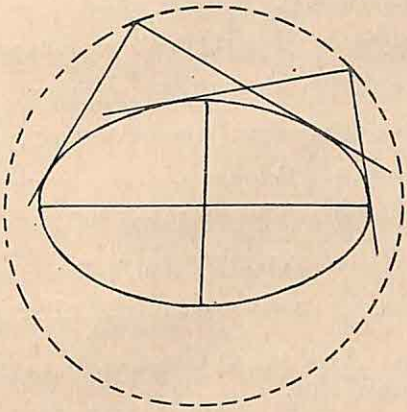
As Q lies on the ellipse, its co-ordinates will satisfy the equation to the ellipse, and hence

$$\frac{x_1^2}{\lambda^2 a^2}+\frac{y_1^2}{\lambda^2 b^2}=1, \quad \text{or,} \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=\lambda^2.$$

Hence, the point P is outside or inside the ellipse according as $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}>1$ or <1.

Ex. 3. *Prove that the locus of point of intersection of any two perpendicular tangents to an ellipse is a circle.*

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ··· (i) be the equation to the ellipse.



Any tangent to (i) is $y=mx+\sqrt{a^2m^2+b^2}$ ··· (ii). For the perpendicular tangent, replacing m by $-\frac{1}{m}$, the equation is $y=-\frac{1}{m}x+\sqrt{\frac{a^2}{m^2}+b^2}$, or, $my=-x+\sqrt{a^2+b^2m^2}$ ··· (iii)

At the point of intersection of (ii) and (iii) both the equations are satisfied. Hence, if we eliminate m between these two equations, the relation obtained will be satisfied at the points of intersection of every such pair of perpendicular tangents, and will thus represent the equation to the desired locus.

Now, from (ii) and (iii),

$$y-mx=\sqrt{a^2m^2+b^2} \quad \text{and} \quad my+x=\sqrt{a^2+b^2m^2}.$$

Squaring and adding,

$$(x^2+y^2)(1+m^2)=(a^2+b^2)(1+m^2).$$

$$\therefore \qquad x^2+y^2=a^2+b^2$$

which evidently represents a circle with its centre at the centre of the ellipse.

Note. This circle is known as the **director circle** of the ellipse.

Ex. 4. *Find the length of the chord of the ellipse* $\frac{x^2}{25}+\frac{y^2}{16}=1$, *whose middle point is* $(\frac{1}{2}, \frac{2}{5})$.

Let the equation to the chord PQ, whose middle point is $(\frac{1}{2}, \frac{2}{5})$ be

$$y-\tfrac{2}{5}=m(x-\tfrac{1}{2}), \text{ or } y=mx+\frac{4-5m}{10}. \quad \cdots \quad \cdots \text{ (i)}$$

The abscissæ of its points of intersection P and Q with the ellipse

$$\frac{x^2}{25}+\frac{y^2}{16}=1 \quad \cdots \quad \cdots \text{ (ii)}$$

are given by [eliminating y between (i) and (ii)]

$$\frac{x^2}{25}+\frac{1}{16}\left\{mx+\frac{4-5m}{10}\right\}^2=1,$$

$$\text{or, } (16+25m^2)x^2+5m(4-5m)x+\frac{(4-5m)^2-1600}{4}=0. \quad \cdots \text{ (iii)}$$

Now, if (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q, then x_1, x_2 are the roots of (iii).

$$\text{Hence, } x_1+x_2=\frac{5m(5m-4)}{16+25m^2} \quad \cdots \quad \cdots \text{ (iv)}$$

$$\text{and} \quad x_1x_2=\frac{(4-5m)^2-1600}{4(16+25m^2)}. \quad \cdots \quad \cdots \text{ (v)}$$

But the abscissa of the middle point of PQ is given to be $\frac{1}{2}$.

$\therefore \ \frac{1}{2}(x_1+x_2)=\frac{1}{2}$, or, $x_1+x_2=1$.

$\therefore$ from (iv), $16+25m^2=5m(5m-4)$. $\therefore \ m=-\frac{4}{5}$.

$$\therefore \text{ (v) gives } x_1x_2=\frac{64-1600}{4.32}=-12.$$

$$\therefore \ (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=1+48=49. \quad \cdots \quad \text{(vi)}$$

As P and Q both lie on (i),

$$y_1-\tfrac{2}{5}=m(x_1-\tfrac{1}{2}),\ y_2-\tfrac{2}{5}=m(x_2-\tfrac{1}{2}).$$

$$\therefore\quad y_1-y_2=m(x_1-x_2)=-\tfrac{4}{5}(x_1-x_2).$$

$$\therefore\quad PQ=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{(x_1-x_2)^2(1+\tfrac{16}{25})}$$
$$=\sqrt{49\times\tfrac{41}{25}}=\tfrac{7}{5}\sqrt{41}.$$

Ex. 5. *Prove that in the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *if the line* $y=m'x$ *bisects all chords parallel to* $y=mx$, *then* $y=mx$ *also bisects all chords parallel to* $y=m'x$.

As in § 7·9, we see that the bisector of all chords parallel to $y=mx$ is the diameter $y=-\dfrac{b^2}{a^2m}x$. Hence, if this diameter be given to be $y=m'x$, we must have $m'=-\dfrac{b^2}{a^2m}$ or $mm'=-\dfrac{b^2}{a^2}$ ··· (i), which is the condition that $y=m'x$ may bisect all chords parallel to $y=mx$.

Similarly, the condition that $y=mx$ may bisect all chords of the ellipse parallel to $y=m'x$ is $m'm=-\dfrac{b^2}{a^2}$, which is identical with (i).

Hence, if $y=m'x$ bisects all chords parallel to $y=mx$ of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, then $y=mx$ will also bisect all chords parallel to $y=m'x$, the common condition being $mm'=-\dfrac{b^2}{a^2}$.

Hence, if a diameter of an ellipse bisects all chords parallel to another diameter, the latter will also bisect all chords parallel to the former.

Two such diameters are referred to as **conjugate diameters** of the ellipse.

Examples VII

1. (i) Find out the eccentricity, and the co-ordinates of the foci of the ellipse $9x^2+25y^2=225$. [*H. S. 1960*]

(ii) Find the co-ordinates of the foci of the ellipse $9x^2+5y^2=45$.

2. An ellipse has its major axis along the x-axis and minor axis along the y-axis. Its eccentricity is $\frac{1}{2}$ and the distance between the foci is 4. Find its equation and show that the ellipse passes through the point $(2, 3)$.

[*H. S. 1961 Compartmental*]

3. (i) Find the equation to the ellipse whose centre is the origin, whose axes are the axes of co-ordinates, and which passes through the points $(-3, \frac{16}{5})$ and $(0, -4)$.

Find also the co-ordinates of its foci.

(ii) An ellipse having centre as origin and axes along the co-ordinate axes, passes through the points $(\frac{3}{2}, -3)$ and $(-\sqrt{6}, 2)$. Find the equations to its directrices.

4. Find the equation to the ellipse having centre as origin, and axes along the axes of co-ordinates, whose latus rectum is 6 and eccentricity $\frac{1}{2}$. Write down the co-ordinates of the extremities of its minor axis.

5. (i) The latus rectum of an ellipse is half its major axis. Find its eccentricity.

(ii) The distance between the focus and directrix of an ellipse is 16 inches and its eccentricity is $\frac{3}{5}$. Obtain the lengths of its principal axes.

6. Find the equation to the ellipse whose focus is $(-1, 1)$, eccentricity is $\frac{1}{2}$, and directrix is $x - y + 3 = 0$.

7. Find the latus rectum, eccentricity and co-ordinates of the centre and foci of the ellipse :

(i) $3x^2 + 4y^2 + 6x - 8y = 5$.

(ii) $9x^2 + 5y^2 - 30y = 0$.

8. Is the point (i) $(2, -1\frac{1}{2})$ (ii) $(2, -1)$, inside or outside the ellipse $4x^2 + 9y^2 = 36$?

9. Find the equation to the tangent of the ellipse $9x^2 + 16y^2 = 144$ having equal positive intercepts on the axes.

[*H. S. 1961*]

10. Find the distance from the origin of the point where the tangent at the extremity of a latus rectum of the ellipse $9x^2 + 25y^2 = 225$ intersects the major axis.

[*H. S. 1960*]

11. Show that $x - 3y = 13$ touches the ellipse.

$$\frac{x^2}{25} + \frac{y^2}{16} = 1. \quad [\textit{ H. S. 1960 Compartmental }]$$

What are the co-ordinates of the point of contact?

12. Find the equations to the tangents to the ellipse $9x^2 + 16y^2 = 36$ which are parallel to $3x - 3y + 7 = 0$, and find out the points of contact.

13. If a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes intercepts α and β along the axes of lengths α and β, prove that $a^2/\alpha^2 + b^2/\beta^2 = 1$.

14. Prove that the product of the perpendiculars from the foci on any tangent to an ellipse is constant, and is equal to the square on the semi-minor axis.

15. The straight line $3x - 5y + 25 = 0$ touches an ellipse whose principal axes are along the axes of co-ordinates, and whose eccentricity in given to be $\frac{3}{5}$. Find the distance between the foci of the ellipse.

16. Find out the equation to the normal to the ellipse $2x^2 + 7y^2 = 71$ at $(2, -3)$ and determine the distance of the point where it intersects the major axis, from the foot of the ordinate.

17. Write down the equation the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at an extremity of the latus rectum, and show that if it passes through an extremity of the minor axis, the eccentricity of the ellipse is given by $e^2 = \frac{1}{2}(\sqrt{5} - 1)$.

18. If the normal to the ellipse $x^2 + 3y^2 = 12$ at a point be inclined at $60°$ to the major axis, show that the line

joining the centre to the point is inclined at 30° to the same axis.

19. Obtain the equation to the chord of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ which is bisected at the point $(2, -1)$.

20. Find the length of the chord of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ intercepted by the line $x+y=3$. What are the co-ordinates of its middle point ?

21. Find the equation to the diameter of the ellipse $6x^2+9y^2=1$ bisecting all chords parallel to $y=x$.

22. Show that the straight lines $3y=4x$ and $x+3y=0$ each bisects all chords of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ parallel to the other.

ANSWERS

1. (i) $\frac{4}{5}$; $(\pm 4, 0)$. (ii) $(0, \pm 2)$. **2.** $\frac{x^2}{16}+\frac{y^2}{12}=1$.

3. (i) $\frac{x^2}{25}+\frac{y^2}{16}=1$; $(\pm 3, 0)$, (ii) $y=\pm 4\sqrt{3}$. **4.** $\frac{x^2}{16}+\frac{x^2}{12}=1$; $(0, \pm 2\sqrt{3})$.

5. (i) $\frac{1}{\sqrt{2}}$. (ii) 30 inches, 24 inches.

6. $8(\{x+1)^2+(y-1)^2\}=(x-y+3)^2$,
or, $7x^2+2xy+7y^2+10x-10y+7=0$.

7. (i) 3 ; $\frac{1}{2}$; $(-1, 1)$; $(0, 1)$ and $(-2, 1)$. (ii) $3\frac{1}{3}$; $\frac{2}{3}$; $(0, 3)$; $(0, 1)$ and $(0, 5)$. **8.** (i) Outside. (ii) Inside. **9.** $x+y=5$. **10.** $6\frac{1}{4}$.
11. $(\frac{16}{15}, -\frac{48}{15})$. **12.** $2x-2y=\pm 5$; $(\frac{8}{5}, -\frac{9}{10})$ and $(-\frac{8}{5}, \frac{9}{10})$.
15. 6. **16.** $21x+4y=30$; $-\frac{4}{7}$. **17.** $x=e(y+ae^2)$.
19. $8x-9y=25$. **20.** $7\frac{33}{41}$; $(\frac{75}{41}, \frac{48}{41})$. **21.** $2x+3y=0$.

CHAPTER VIII

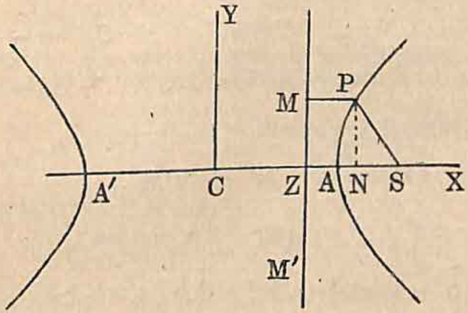
HYPERBOLA

8·1. Hyperbola.

A hyperbola is a curve traced out by a point which moves on a plane so that its distance from a fixed point on the plane always bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the ratio being greater than unity.

The fixed point is called the *focus*, the fixed straight line is called the *directrix*, and the constant ratio (which is greater than unity in this case) is called the *eccentricity* of the hyperbola.

8·2. Standard equation of a hyperbola.



Let S be the focus, MM' the directrix, and $e(>1)$ the given eccentricity of the hyperbola.

Draw SZ perpendicular from S on MM', and let it be divided internally at A and externally at A' in the ratio $e:1$. As $e>1$, $SA'>A'Z$, and accordingly A' is to the left of S as in the figure, on the side of the directrix MZM' opposite to A, S being not between A and A'.

Then, $SA = e.AZ$ and $SA' = e.A'Z$.

Hence, by definition of the hyperbola, A and A' are points on the hyperbola.

Let C be the middle point of AA'.

Thus, $SA + SA' = e(AZ + A'Z)$

and $SA' - SA = e(A'Z - AZ)$,

or, $2CS = e.AA' = e.2CA$ and AA' or $2CA = e.2CZ$.

Let $CA = CA' = a$. Thus, $CS = ae$, and $CZ = \dfrac{a}{e}$.

Let us choose C as origin, and CX along $A'A$ as x-axis, the y-axis CY being parallel to $M'M$ *i.e.*, perpendicular to $A'A$ through C.

Now, P being any point on the hyperbola, whose co-ordinates are (x, y), let PN be the perpendicular from P to the x-axis, and PM be perpendicular to the directrix MM'. Then, $CN = x$, $PM = ZN = CN - CZ = x - \dfrac{a}{e}$. Also co-ordinates of S are evidently ae, 0. ($\because$ $CS = ae$).

Hence, from the property of the hyperbola,

$$SP = e.PM, \quad \text{or} \quad SP^2 = e^2 PM^2.$$

$$\therefore \quad (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

or, $x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$, ($\because$ $e > 1$ here)

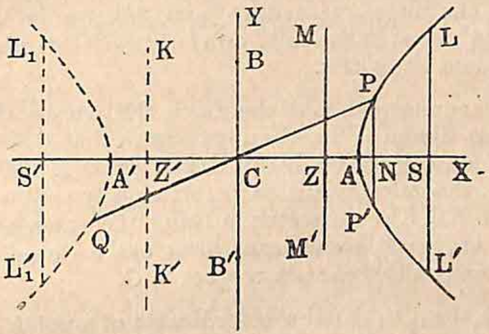
or, writing $a^2(e^2 - 1) = b^2$,

$$\frac{\mathbf{x}^2}{\mathbf{a}^2} - \frac{\mathbf{y}^2}{\mathbf{b}^2} = \mathbf{1}. \qquad \cdots \qquad \cdots \quad \text{(i)}$$

This being the relation satisfied by the co-ordinates of any point on the hyperbola, it represents the equation to the hyperbola in its standard form.

Here C, the middle point of AA' (called the *centre*) is the origin, $CA = CA' = a$, and $b^2 = a^2(e^2 - 1)$.

8·3. Shape and elementary properties of the hyperbola.



From the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of the hyperbola, the following points may be noted :

If $y=0$, $x=\pm a$, so that the hyperbola cuts the x-axis at points A and A' given by $x=a$ and $x=-a$ respectively.

If $x=0$, y^2 is negative and so y is imaginary. Accordingly the curve does not cut the y-axis at all.

For values of $x < a$ or $> -a$ (*i.e.*, within AA'), $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$ is negative, and so y is imaginary. Hence, there is no portion of the hyperbola within the range AA'.

For values of $x > a$ or $< -a$, $\frac{x^2}{a^2} > 1$ and so $\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$ is positive. Hence, y has two equal and opposite values. Thus from A to the right, or from A' to the left, the curve extends, being symmetrical with respect to the x-axis, y having greater and greater magnitudes as the magnitude of x becomes greater and greater.

Again for any values of y, $\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2}$ is positive, and so

x has two equal and opposite values. Hence, the curve is symmetrical with respect to the y-axis.

Thus, *the hyperbola consists of two detached portions* as shown in the figure, extending from A towards the right and from A' towards the left, *being symmetrical about both the x-axis and the y-axis.*

From symmetry about the y-axis CY, we see that if we take points S' and Z' on the x-axis such that $CS' = CS$ and $CZ' = CZ$ on opposite sides of C, and draw $KZ'K'$ parallel to MZM', the curve might be drawn equally well with S' as focus and $KZ'K'$ as directrix, e being the same as before. Hence, *a hyperbola has a second focus and a second directrix* symmetrically situated with respect to C.

Lastly, if x_1, y_1 be the co-ordinates of a point P on the hyperbola so that they satisfy the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the co-ordinates $-x_1$, $-y_1$ will also satisfy it, and accordingly the diagonally opposite point Q, where PQ is bisected at C, is also a point on the hyperbola. Thus, *every chord of the hyperbola through C is bisected at C*, and so the hyperbola is symmetrical with respect to the origin C, the mid-point of AA'. This is why C is called the *centre* of the hyperbola.

The x-axis is here referred to as the *transverse axis*, and the length $AA' = 2a$ is called the length of the transverse axis. The y-axis here is referred to as the *conjugate axis*, and a length $BB' = 2b$ (where $CB = CB' = b$) along this axis is referred to as the length of the conjugate axis.

The chord LSL' through the focus S (or $L_1S'L'_1$ through the focus S') perpendicular to the transverse axis (*i.e.*, parallel to the directrix) is called the *latus rectum* of the hyperbola.

Now, ae being the length CS, the x-co-ordinate of the extremity L of the latus rectum is ae. Hence, from the equation to the hyperbola, the y-co-ordinate of L is given by

$$\frac{a^2e^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Hence, $y = \pm b\sqrt{e^2-1} = \pm a(e^2-1)$.

Thus, *the length LL′ of the latus rectum*

$$= \mathbf{2a(e^2-1) = 2\frac{b^2}{a}}.$$

$\therefore$ Semi-latus rectum $= \dfrac{b^2}{a} = a(e^2-1)$.

Co-ordinates of the extremity L of the latus rectum are

$$ae,\ a(e^2-1).$$

The eccentricity of the hyperbola is given by

$$\mathbf{b^2 = a^2(e^2-1)} \text{ or } \mathbf{e^2 = \frac{a^2+b^2}{a^2}}.$$

Note 1. If $a=b$, the hyperbola is said to be a **rectangular** or **equilateral hyperbola.** For a rectangular hyperbola the eccentricity

$$e = \sqrt{2}.$$

Note 2. Lengths of the focal distances SP, S′P of any point P on the hyperbola :

Let the co-ordinates of P be (x_1, y_1). Those of S being $(ae, 0)$ we get

$$SP^2 = (x_1-ae)^2 + y_1^2 = (x_1-ae)^2 + b^2\left(\frac{x_1^2}{a^2}-1\right)$$

[*from the equation to the hyperbola*]

$$= (x_1-ae)^2 + (e^2-1)(x_1^2-a^2) \qquad [\because\ b^2 = a^2(e^2-1)]$$

$$= e^2x_1^2 - 2x_1ae + a^2 = (ex_1-a)^2.$$

$\therefore$ $\mathbf{SP = ex_1 - a}$, which is the positive value of SP,

$\because$ $x_1 > a$ and $e > 1$ here.

Similarly, $\mathbf{S'P = ex_1 + a}$.

Thus, $S'P - SP = 2a =$ length of the transverse axis.

Hence, we get the important property of the hyperbola, namely, *the difference of the focal distances of any point on the hyperbola is constant and equal to the transverse axis.*

8·4. Equation to the tangent at a given point x_1, y_1 on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Let P be the point x_1, y_1 on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \ldots \quad \ldots \quad \text{(i)}$$

and let $Q(x_2, y_2)$ be a neighbouring point on it. The equation to the chord PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad \ldots \quad \ldots \quad \text{(ii)}$$

Now, since P and Q both lie on the hyperbola (i), we have

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1 \quad \cdots \quad \text{(iii)} \quad \frac{x_2^2}{a^2}-\frac{y_2^2}{b^2}=1 \quad \ldots \quad \text{(iv)}$$

Hence, subtracting,

$$\frac{x_2^2-x_1^2}{a^2}-\frac{y_2^2-y_1^2}{b^2}=0, \text{ or } \frac{y_2-y_1}{x_2-x_1}=\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}.$$

$\therefore$ equation (ii) can be written as

$$y-y_1=\frac{b^2}{a^2}\cdot\frac{x_2+x_1}{y_2+y_1}(x-x_1) \quad \ldots \quad \text{(v)}$$

Now, make Q approach P and ultimately coincide with it, so that the co-ordinates (x_2, y_2) coincide with (x_1, y_1). In that limiting position the straight line PQ becomes the tangent at P, whose equation [*from (v)*] then becomes

$$y-y_1=\frac{b^2}{a^2}\cdot\frac{x_1}{y_1}(x-x_1),$$

or, $$\frac{y_1}{b^2}(y-y_1)=\frac{x_1}{a^2}(x-x_1),$$

or, $$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1. \quad [\textit{by (iii)}]$$

Hence, the equation to the tangent at x_1, y_1 to the hyperbola (i) is

$$\frac{\mathbf{xx_1}}{\mathbf{a^2}}-\frac{\mathbf{yy_1}}{\mathbf{b^2}}=1.$$

8·5. Equation to the normal at $\mathbf{x_1, y_1}$ to the hyperbola $\frac{\mathbf{x^2}}{\mathbf{a^2}}-\frac{\mathbf{y^2}}{\mathbf{b^2}}=1$.

The tangent at (x_1, y_1) to the hyperbola is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$,

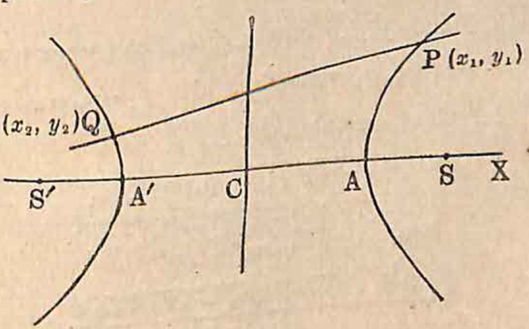
or, $y=\frac{b^2x_1}{a^2y_1}x-\frac{b^2}{y_1}$, of which the '$m$' is $\frac{b^2x_1}{a^2y_1}$.

The normal which is perpendicular to the tangent through x_1, y_1, has its 'm' $=-\frac{a^2y_1}{b^2x_1}$, and accordingly its equation is

$$y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1),$$

or,
$$\frac{\mathbf{x-x_1}}{\frac{\mathbf{x_1}}{\mathbf{a^2}}}=\frac{\mathbf{y-y_1}}{-\frac{\mathbf{y_1}}{\mathbf{b^2}}}.$$

8·6. Length of the chord of the hyperbola $\frac{\mathbf{x^2}}{\mathbf{a^2}}-\frac{\mathbf{y^2}}{\mathbf{b^2}}=1$, intercepted by the straight line $\mathbf{y=mx+c}$.



At the points of intersection of the line with the hyperbola, both the equations are satisfied. Hence, eliminating y between the two equations, the abscissæ of the points of intersection will be given by

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1,$$

or, $$(a^2m^2 - b^2)x^2 + 2mca^2x + a^2(b^2+c^2) = 0 \quad \ldots \quad \text{(i)}$$

which being a quadratic in x, there are only two values of x and accordingly only two points of intersection of the given straight line with the hyperbola (real, coincident or imaginary).

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the two points P and Q of intersection. Then, x_1 and x_2 are the roots of (i).

$$\therefore \quad x_1 + x_2 = -\frac{2mca^2}{a^2m^2 - b^2} \text{ and } x_1x_2 = \frac{a^2(b^2+c^2)}{a^2m^2-b^2}.$$

$$\therefore \quad (x_1 - x_2)^2 = (x_1+x_2)^2 - 4x_1x_2$$

$$= \frac{4m^2c^2a^4}{(a^2m^2-b^2)^2} - \frac{4a^2(b^2+c^2)}{a^2m^2-b^2}$$

$$= \frac{4a^2\{m^2c^2a^2 - (b^2+c^2)(a^2m^2-b^2)\}}{(a^2m^2-b^2)^2}$$

$$= \frac{4a^2b^2(c^2 - a^2m^2 + b^2)}{(a^2m^2-b^2)^2}.$$

Again, P and Q lying on the given line $y = mx + c$,

$$y_1 = mx_1 + c,\ y_2 = mx_2 + c. \quad \therefore \quad y_1 - y_2 = m(x_1 - x_2).$$

$\therefore$ length of the chord PQ

$$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} = \sqrt{(x_1-x_2)^2(1+m^2)}$$

$$= \sqrt{\frac{4a^2b^2(c^2-a^2m^2+b^2)(1+m^2)}{(a^2m^2-b^2)^2}}$$

$$= \frac{\mathbf{2ab\sqrt{1+m^2}\sqrt{c^2-a^2m^2+b^2}}}{\mathbf{a^2m^2-b^2}}.$$

Cor. Condition of tangency.

The given line will touch the hyperbola only when the two points of intersection come into coincidence, *i.e.*, when the length of the chord intercepted is zero. Hence, the condition that the given line $y=mx+c$ may touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$c^2-a^2m^2+b^2=0, \quad \text{or,} \quad \mathbf{c=\pm\sqrt{a^2m^2-b^2}}.$$

8·7. *To show that $y=mx+\sqrt{a^2m^2-b^2}$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for all values of m, and to find the point of contact.*

The tangent at x_1, y_1 of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1, \quad \text{or,} \quad y=\frac{b^2x_1}{a^2y_1}x-\frac{b^2}{y_1} \qquad \ldots \quad \text{(i)}$$

If the line $y=mx+\sqrt{a^2m^2-b^2}$ ··· (ii) be a tangent to the hyperbola at (x_1, y_1), the equations (i) and (ii) must be identical. Hence, comparing coefficients,

$$\frac{b^2x_1}{a^2y_1}=m, \quad -\frac{b^2}{y_1}=\sqrt{a^2m^2-b^2};$$

$$\therefore \quad y_1=-\frac{b^2}{\sqrt{a^2m^2-b^2}}, \quad x_1=\frac{ma^2y_1}{b^2}=-\frac{ma^2}{\sqrt{a^2m^2-b^2}}.$$

The line (ii) therefore will touch the hyperbola only if the assumed point x_1, y_1 is really a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *i.e.*, if

$$\left(-\frac{am}{\sqrt{a^2m^2-b^2}}\right)^2-\left(\frac{-b}{\sqrt{a^2m^2-b^2}}\right)^2=1$$

which is evidently satisfied.

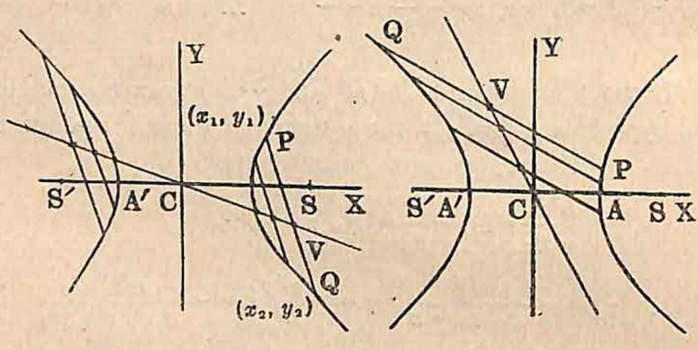
Thus, $y = mx + \sqrt{a^2m^2 - b^2}$ is always a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whatever m may be, and the point of contact is given by

$$\mathbf{x_1 = -\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \quad y_1 = -\frac{b^2}{\sqrt{a^2m^2 - b^2}}.}$$

Similarly, $y = mx - \sqrt{a^2m^2 - b^2}$ is also a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, for all values of m, the co-ordinates of the point of contact being

$$\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{b^2}{\sqrt{a^2m^2 - b^2}}.$$

8·8. Locus of the middle points of a system of parallel chords ; diameter.



Let PQ, given by the equation

$$y = mx + c \qquad \dots \quad \text{(i)}$$

be anyone of a system of parallel chords of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \dots \quad \text{(ii)}$$

As the chords are parallel, m is the same for all chords, but c is different for different chords of the system.

At the common points of intersection of (i) and (ii), eliminating y, the abscissæ are given by the roots of the equation

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1,$$

or, $$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(b^2 + c^2) = 0. \quad \cdots \quad \text{(iii)}$$

Thus, if (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q, x_1, x_2 are the roots of (iii), and so.

$$x_1 + x_2 = -\frac{2a^2mc}{a^2m^2 - b^2}.$$

Hence, if X, Y be the co-ordinates of the mid-point V of PQ,

$$X = \tfrac{1}{2}(x_1 + x_2) = -\frac{a^2mc}{a^2m^2 - b^2}.$$

Also, $\because$ V is a point on (i), $Y = mX + c$.

$\therefore$ eliminating c,

$$X = \frac{-a^2m(Y - mX)}{a^2m^2 - b^2}, \quad \text{or,} \quad -b^2X = -a^2mY,$$

or, $Y = \dfrac{b^2}{a^2m}X$, which being independent of c holds for the middle point of any chord of the parallel system. Hence, the locus of the middle points of a system of parallel chords of the hyperbola, parallel to $y = mx$, is

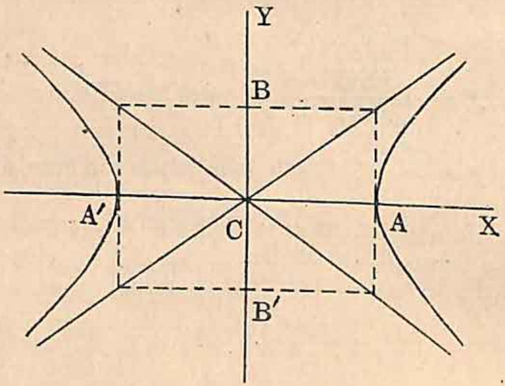
$$\mathbf{y = \frac{b^2}{a^2m}x}$$

which is evidently a straight line passing through the origin *i.e.*, the centre C of the hyperbola. This straight line is called a *diameter* of the hyperbola. For different values of m (*i.e.*, for differently directed system of parallel chords) we get different diameters, all passing through the centre.

8·9. Asymptotes of a hyperbola.

We have noticed in § 8·7 that the line given by $y = mx + \sqrt{a^2m^2 - b^2}$ is always a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, and the co-ordinates of the point of contact are $-\frac{a^2m}{\sqrt{a^2m^2-b^2}}, -\frac{b^2}{\sqrt{a^2m^2-b^2}}$. Now, if m be so chosen that $a^2m^2 - b^2 = 0$, or $m = \pm \frac{b}{a}$, the co-ordinates of the point of contact both tend to infinity.

Hence, the straight lines $y = \pm \frac{b}{a}x$ are both tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the points of contact tend to infinity. These lines are defined as *asymptotes* to the hyperbola.



They are inclined to the transverse axis at an angle θ, where $\tan\theta = \pm \frac{b}{a}$. Hence, with sides equal and parallel to the transverse axis $2a$ and conjugate axis $2b$ of the hyperbola, if we construct a rectangle with centre at the

origin, the diagonals will be the asymptotes, which will continually approach the hyperbola, and will ultimately touch it at infinite distance.

In the particular case when $b=a$, the asymptotes are inclined to the x-axis at angles $\pm 45^\circ$, and so they are mutually perpendicular. The hyperbola in this case, when its transverse and conjugate axes are equal in length, is defined to be a *rectangular or equilateral hyperbola*, having its asymptotes mutually perpendicular.

8·10. Illustrative Examples.

Ex. 1. *The co-ordinates of the foci of a hyperbola are* $(-5, 3)$ *and* $(7, 3)$, *and its eccentricity is* $\frac{3}{2}$. *Find its equation and determine the length of its latus rectum.*

Let $S\,(7, 3)$ and $S'\,(-5, 3)$ be the given foci. The eccentricity $e=\frac{3}{2}$. If $2a$ be the length of the transverse axis, then $SS'=2ae$,

$$\text{or,}\quad 12=2a\times\tfrac{3}{2}. \qquad \therefore\ a=4.$$

Also, $2b$ being the conjugate axis,

$$b^2=a^2(e^2-1)=16\,(\tfrac{9}{4}-1)=20.$$

Hence, the length of the latus rectum

$$=2\,\frac{b^2}{a}=2\cdot\frac{20}{4}=10.$$

Again, the middle point C of SS' is clearly the centre, and its co-ordinates are

$$\tfrac{1}{2}\,(7-5) \text{ and } \tfrac{1}{2}\,(3+3),\ \ i.e.,\ (1, 3).$$

Also, the transverse axis, which is along the line SS' has equation

$$(y-3)(7+5)=(x-7)(3-3)=0,\quad i.e.,\quad y=3$$

and hence it is parallel to the x-axis.

Now referred to the centre C as origin, and transverse axis as x-axis, the equation to the hyperbola (whose $a^2=16$ and $b^2=20$) is evidently

$$\frac{x^2}{16}-\frac{y^2}{20}=1.$$

Hence, referred to given axes [with reference to which C has co-ordinaies $(1, 3)$, and to which the transverse and conjugate axes

of the hyperbola are parallel] the required equation to the hyperbola is evidently

$$\frac{(x-1)^2}{16}-\frac{(y-3)^2}{20}=1. \quad \cdots \quad \cdots \quad \text{(i)}$$

Alternatively

Since the difference of the focal distances of any point on the hyperbola is equal to its transverse axis, which is $2a=8$ here, if (x, y) be the co-ordinates of any point on the hyperbola,

$$\sqrt{(x+5)^2+(y-3)^2} \sim \sqrt{(x-7)^2+(y-3)^2}=8,$$

or, $$\sqrt{(x+5)^2+(y-3)^2}=\sqrt{(x-7)^2+(y-3)^2}\pm 8.$$

Hence, squaring and transposing,

$$24x-88=\pm 16\sqrt{(x-7)^2+(y-3)^2},$$

or, $$(3x-11)^2=4\{(x-7)^2+(y-3)^2\},$$

or, $$5x^2-4y^2-10x+24y-111=0$$

which is the required equation to the hyperbola, and is the same as equation (i) already obtained above.

Ex. 2. *Prove that the tangent to the hyperbola $x^2-3y^2=12$ at the point $(-6, 2\sqrt{2})$ bisects the angle between the focal distances of the point.*

The given equation to the hyperbola can be written in the form

$$\frac{x^2}{12}-\frac{y^2}{4}=1 \quad \cdots \quad \cdots \quad \cdots \quad \text{(i)}$$

and hence the co-ordinates of its foci S and S' are easily seen to be

$$\pm\sqrt{12+4},\ 0 \text{ i.e., } \pm 4,\ 0.$$

Now P being the point $(-6, 2\sqrt{2})$ on the hyperbola, the equations to the focal distances SP and $S'P$ are

$$y=\frac{2\sqrt{2}}{-6-4}(x-4), \text{ i.e., } x\sqrt{2}+5y-4\sqrt{2}=0 \cdots \quad \text{(ii)}$$

and $$y=\frac{2\sqrt{2}}{-6+4}(x+4), \text{ i.e., } x\sqrt{2}+y+4\sqrt{2}=0 \quad \cdots \quad \text{(iii)}$$

respectively.

The equation to the bisector of the angle SPS', *i.e.*, between (ii) and (iii), in which the origin lies, is

$$\frac{x\sqrt{2}+5y-4\sqrt{2}}{-\sqrt{2+25}}=\frac{x\sqrt{2}+y+4\sqrt{2}}{\sqrt{2+1}}$$

or, $\quad x\sqrt{2}+5y-4\sqrt{2}+3(x\sqrt{2}+y+4\sqrt{2})=0,$

or, $\quad x+\sqrt{2}y+2=0 \qquad \cdots \qquad \cdots \qquad \cdots \quad \text{(iv)}$

Now equation to the tangent to (i) at $(-6, 2\sqrt{2})$ is

$$\frac{x.(-6)}{12}-\frac{y.(2\sqrt{2})}{4}=1,$$

or, $\quad x+\sqrt{2}y+2=0$, the same as (iv). Hence, the tangent at the point is the bisector of the angle between the focal distances of the point.

Ex. 3. *Find the length of the chord of the hyperbola $x^2-4y^2=9$ along the straight line $x+4y+3=0$, and determine the co-ordinates of its middle point.*

At the points of intersection of the hyperbola $x^2-4y^2=9 \cdots$ (i) with the straight line $x+4y+3=0 \cdots$ (ii), eliminating x, the ordinates are the roots of

$$(4y+3)^2-4y^2=9, \quad \text{or,} \quad y(y+2)=0.$$

$\therefore \; y=0$, or -2. The corresponding values of x from (ii) are $x=-3$ or 5.

Hence, the co-ordinates of the extremities of the chord are $(-3, 0)$ and $(5, -2)$.

Thus, the length of the chord $=\sqrt{(-3-5)^2+(0+2)^2}=2\sqrt{17}$.

Also, the co-ordinates of the middle point of the chord are $\frac{1}{2}(-3+5)$, $\frac{1}{2}(0-2)$ *i.e.*, $1, -1$.

Ex. 4. *Prove that the portion of the tangent at any point of a hyperbola intercepted between the asymptotes is bisected at the point of contact.*

Let $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ...... (i) be the equation to a hyperbola. Its asymptotes are given by $y=\dfrac{b}{a}x$ ...... (ii) and $y=-\dfrac{b}{a}x$ ...... (iii).

The tangent at any point $P(x', y')$ to the hyperbola (i) is $\frac{xx'}{a^2}-\frac{yy'}{b^2}=1$ ··· (iv). This meets (ii) at a point Q whose x co-ordinate [by eliminating y between (ii) and (iv)] is given by

$$\frac{xx'}{a^2}-\frac{y'}{a^2}\cdot\frac{b}{a}x=1, \quad \text{or} \quad x=\frac{a}{\frac{x'}{a}-\frac{y'}{b}}.$$

Similarly (iv) meets (iii) at R whose x-co-ordinate is given by

$$x=\frac{a}{\frac{x'}{a}+\frac{y'}{a}}.$$

The x-co-ordinate of the mid-point of QR is

$$\frac{1}{2}\left[\frac{a}{\frac{x'}{a}-\frac{y'}{b}}+\frac{a}{\frac{x'}{a}+\frac{y'}{b}}\right]=\frac{x'}{\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}=x \quad [\,by\ (i)\,].$$

Similarly, the y-co-ordinate of the mid-point of QR is y'. Thus, P is the mid-point of QR.

Examples VIII

1. Obtain the equation to the hyperbola whose focus is $(a, 0)$, directrix is the straight line $x=\frac{1}{2}a$, and eccentricity is $\sqrt{2}$. [*H. S. 1960*]

2. Find the equation to the hyperbola, referred to its axes as axes of co-ordinates,

(i) whose eccentricity is $\sqrt{2}$, and distance between its foci 16.

(ii) whose latus rectum is $10\frac{2}{3}$ and distance between focus and directrix is $3\frac{1}{5}$.

3. In the hyperbola $4x^2-9y^2=36$, find the lengths of the axes, the co-ordinates of the foci, the eccentricity and the length of the latus rectum. [*H. S. 1961*]

4. A point moves on the plane of the co-ordinate axes so that the difference of its distances from the points $(\pm 3, 0)$ is always 4. Prove that it traces out a hyperbola whose eccentricity and length of latus rectum you are to determine.

5. By transferring the origin suitably, show that the equation $5x^2 - 4y^2 - 20x - 8y - 4 = 0$ represents a hyperbola, and determine its eccentricity, co-ordinates of its foci, and equations to the directrices.

6. Find the co-ordinates of the foci of the hyperbola $x^2 - y^2 = 9$.

Also find the distance from the origin of the point where the tangent to the above hyperbola at $(5, 4)$ meets the x-axis.
[*H. S. 1960, Compartmental*]

7. Show that the tangent to the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ at each of the points (i) $(-5, \frac{9}{4})$, (ii) $(8, -3\sqrt{3})$ bisects the angle between the focal distances of the corresponding point.

8. Find the length intercepted on the conjugate axis between the tangents at the two extremities of a latus rectum of the hyperbola $7x^2 - 9y^2 = 63$.

9. (i) Find the points on the hyperbola $3x^2 - 5y^2 = 15$ at which the tangents are inclined at 60° to the x-axis.

(ii) Find the tangents perpendicular to $x + 2y = 0$ of the hyperbola $7x^2 - 4y^2 = 28$, and find the points of contact.

10. Prove that the locus of the point of intersection of any two perpendicular tangents to a hyperbola is a circle.

11. Find the equation to the normal to the hyperbola $16x^2 - 25y^2 = 31$ at the point whose ordinate is -3 and abscissa positive.

12. In the rectangular hyperbola $x^2 - y^2 = a^2$, show that

(i) the intercept on the x-axis of the normal at any point is double the abscissa of the point.

(ii) the length of the normal at any point intercepted between the axes is bisected at the point.

13. Obtain the length of the chord of the hyperbola $\frac{x^2}{9} - \frac{y^2}{25} = 1$, passing through the origin and making equal angles with the axes. [*H. S. 1960, Compartmental*]

14. Find the equation to the chord of the hyperbola $x^2 - 2y^2 = 1$ which is bisected at the point $(-3, -1)$.

15. Find the length of the chord of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ along the line $3x + 2y = 12$.

16. Find the equation to the diameter of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ bisecting all chords parallel to $x - 2y + 7 = 0$.

17. If P be a point on a rectangular hyperbola, prove that $SP.S'P = CP^2$.

18. The normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes ; prove that the locus of P is the hyperbola

$$a^2x^2 - b^2y^2 = (a^2 + b)^2.$$

ANSWERS

1. $2x^2 - 2y^2 = a^2$. **2.** (i) $x^2 - y^2 = 32$. (ii) $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

3. 6, 4 ; $(\pm\sqrt{13}, 0)$; $\frac{1}{3}\sqrt{13}$; $2\frac{2}{3}$. **4.** $\frac{2}{3}$; 5.

5. $\frac{3}{2}$; $(5, -1)$ and $(-1, -1)$; $x = 3\frac{1}{3}$ and $x = \frac{2}{3}$. **6.** $(\pm 3\sqrt{2}, 0)$; $1\frac{4}{5}$.

8. 6. **9.** (i) $\left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{5}{2}, -\frac{\sqrt{3}}{2}\right)$.

(ii) $y = 2x \pm 3$; $(\frac{8}{3}, \frac{7}{3})$ and $(-\frac{8}{3}, -\frac{7}{3})$. **11.** $75x - 64y = 492$.

13. $\frac{15}{2}\sqrt{2}$. **14.** $3x - 2y + 7 = 0$. **15.** $\frac{4}{3}\sqrt{13}$. **16.** $5x - 2y = 0$.

SOLID GEOMETRY

CHAPTER I

FUNDAMENTAL CONCEPTS AND DEFINITIONS

1·1. We give below definitions and chief characteristics of some fundamental entities used in Solid Geometry.

(i) A **point** has position but no magnitude ; that is, it has neither length nor breadth nor thickness.

(ii) A **line** has length but no breadth and thickness.

(iii) A **surface** has length, breadth but no thickness.

(iv) A **solid** has length, breadth and thickness.

Thus, a brick is a *solid*, one of its six faces is a *surface*, an edge is a *line* and a corner is a *point*.

Each of the three elements (1) length, (2) breadth and (3) thickness of a body is called a *dimension* of the body.

Thus, a point has no (or zero) dimension, a line has one dimension, a surface has two dimensions and a solid has three dimensions.

A solid is bounded by surfaces, a surface is bounded by lines and a line is bounded by points.

Solid Geometry deals with the properties of lines, surfaces and solid in three dimensional space.

When co-ordinates are not used in the treatment of Solid Geometry, it is called **Pure** Solid Geometry and when co-ordinates are used, it is called **Analytical** Solid Geometry.

This present treatise is a Pure Solid Geometry.

1·2. If a surface be such that the straight line joining *any two* points in it lies wholly on the surface, it is called a plane surface or more simply a *plane*.

Note. In this treaties straight lines are supposed to be of infinite length and the planes of infinite extent unless anything to the contrary is stated. The expression *lies wholly on the surface* means that every

point in the straight line however produced both ways lies in the surface.

1·3. Lines are said to be **co-planar** if they be in a plane or a plane can be made to pass through them.

1·4. Two straight lines are said to be *parallel*, when they *lying in the same plane*, do not meet however far they may be produced both ways. It should be noted that every pair of parallel straight lines is co-planar.

1·5. Lines are said to be **skew** when a plane cannot be made to pass through them and they do not meet however far they may be produced.

Thus, it should be noted that skew lines neither intersect nor are they parallel.

From (1·3), (1·4), (1·5), it is clear that two straight lines are either co-planar or skew. If they are co-planar, they either intersect or are parallel and if they are not co-planar *i.e.*, if they are skew, they neither intersect nor are they parallel.

1·6. Planes are said to be **parallel** when they do not meet even if they are indefinitely produced in any direction.

1·7. A **straight line** and a **plane** are said to be **parallel** when they do not meet even if produced indefinitely in any direction.

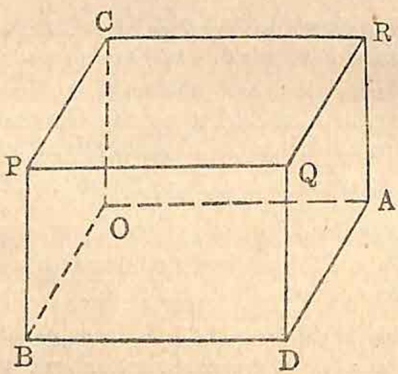


Fig. 1

In the adjoining figure, the lines PQ, BD are parallel ; they lie in the plane $PQDB$. PQ, OB are skew lines. Planes $PQDB$, $OARC$, are parallel. Straight line PQ is parallel to the plane $OARC$.

1·8. A straight line is said to be **perpendicular** to a plane, if it is perpendicular to every straight line which meets it in that plane.

Such a straight line is said to be **normal** to that plane.

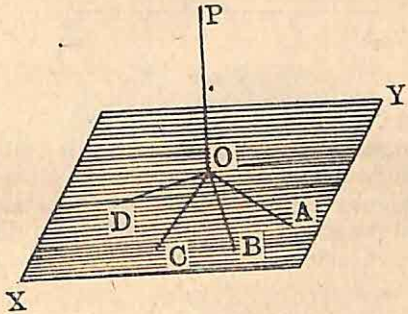


Fig. 2

Thus, PO is said to be perpendicular to the plane XY if it is perpendicular to every straight line OA, OB, OC, OD etc. drawn in the plane to meet it at O.

1·9. If a straight line is parallel to the direction of a plumb-line hanging freely at rest, it is called a **vertical line.** The plane which is perpendicular to the vertical line is called the **horizontal plane.**

Any straight line drawn in the horizontal plane is called a *horizontal* line.

1·10. The *angle between two skew straight lines* (*i.e.*, two non-co-planar straight lines) is measured by the angle contained by one of them and a straight line drawn through any point in that line parallel to the other.

Let AB and CD (Fig. 3) be two skew straight lines ; through any point P on AB draw the straight line PQ parallel to CD. Then $\angle QPB$ is the angle between the skew st. lines AB and CD.

Since, the sides of a triangle all lie in one plane, a triangle is a plane figure ; a parallelogram is also a plane figure, but

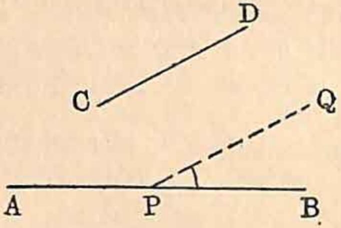


Fig. 3

all the sides of a quadrilateral need not lie in one plane, so a quadrilateral may or may not be a plane figure. If a quadrilateral be drawn such that two of its adjacent sides lie in one plane and the other two in another plane, such a quadrilateral is called **skew** or **gauche.** If the extremities of a pair of finite skew lines be joined, a skew quadrilateral is formed.

That the sides of a quadrilateral need not lie in one plane can be easily seen by folding a plane quadrilateral about either diagonal

1·11. The locus of the feet of the perpendiculars drawn from all points in a line on a given plane is called the *orthogonal projection* (or simply *the projection*) of the line on the plane.

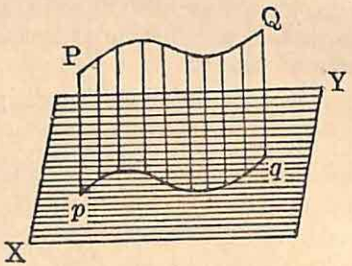


Fig. 4

In the adjoining figure, the projection of the line PQ on the plane XY is the line pq.

The projection of a straight line is itself a straight line. A straight line and its projection are co-planar.

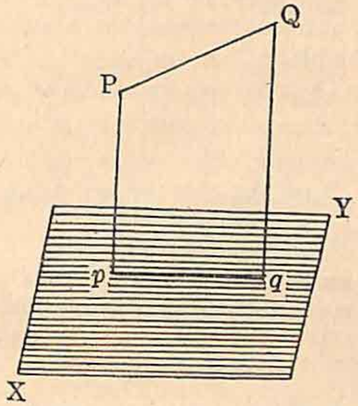


Fig. 5

In the adjoining figure, the straight line pq is the projection of the straight line PQ. PQ, pq are co-planar.

The projection of a curved line on a plane may be a curved line, as in Fig. 4, or it may sometimes be a straight

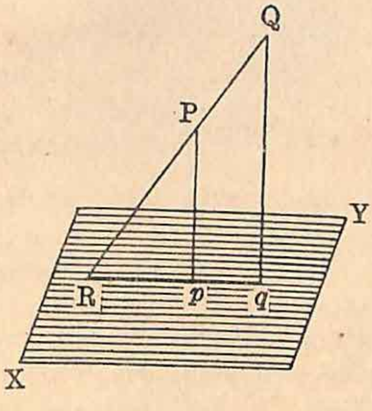


Fig. 6

line when the curved line lies on a plane perpendicular to the plane of projection.

1·12. The *angle between a straight line and a plane* is measured by the angle between the straight line and its projection on the plane.

Let the straight line PQ (Fig. 6) and its projection pq (on the plane XY) produced if necessary meet at the point R in the plane XY (as shown in the adjoining figure).

Then $\angle QRq$ is called the angle between the straight line PQ and the plane XY.

1·13. The *angle between two planes* is measured by the plane angle contained by the two straight lines drawn from *any* point in the line of section of the two planes, perpendicular to that line of section, one in each plane.

This angle is called a **dihedral angle** between the two planes.

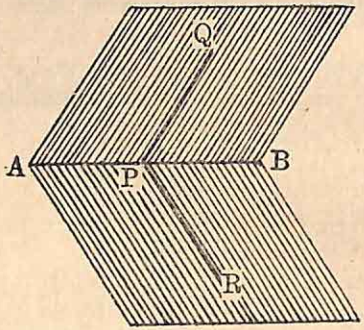


Fig. 7

Let π_1, π_2 be two planes and let AB be their line of intersection. From *any* point P in AB, draw two straight lines PQ, PR both perpendicular to AB, one in each plane. Then the angle between the two planes is measured by the angle QPR, which is called the dihedral angle between the two planes.

CHAPTER II

AXIOMS AND THEOREMS

Axioms.—The following fundamental properties have been laid down as axioms in the Pre-University syllabus of the Calcutta University.

(1) One and only one plane passes through a given straight line and a given point outside it.

(2) If two planes have one point in common, they have at least a second point common.

In the syllabus of the Higher Secondary Course, the following have been laid down as axioms :

(1) One and only one plane may be made to pass through any two intersecting straight lines.

(2) Two intersecting planes cut one another in a straight line and in no point outside it.

From above, the following inferences can be drawn at once :

The position of a plane is fixed if it passes through

(1) a given straight line and a point outside it.

(2) two intersecting straight lines.

(3) three non-collinear points,

(4) two parallel straight lines.

The axiom (2) of the Higher Secondary Course can be deduced as a theorem from axiom (2) of the Pre-University Course as shown below :

THEOREM I

Two intersecting planes cut one another in a straight line, and in no point outside it.

Let MN, XY be two intersecting planes. It is required to prove that the planes MN and XY intersect in a straight line and in no point outside it.

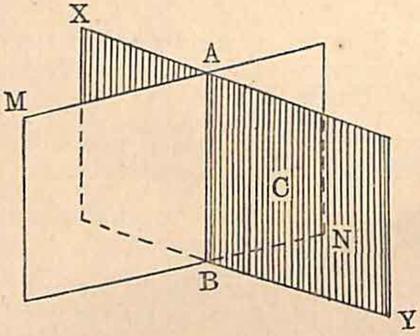


Fig. 8

Since the two planes intersect, let A be a point of intersection *i.e.*, A is a point common to both the planes. Hence, by axiom (2), there is at least a second point, say B, common to both.

Then the straight line joining AB lies wholly in both the planes *i.e.*, the two planes intersect along the straight line AB. If the two planes have in common a point C, which is outside the straight line AB, then they each coincide with the plane of the triangle ABC and therefore with one another, which is not the case here.

Thus, no point outside AB is common to both. Hence, the two planes MN, XY cut one another in AB and in no point outside it. Q. E. D.

Exercises 1

1. If of any three straight lines, each pair cut one another, show that they must be co-planar.

2. The lines of intersection of two parallel planes with any third plane are parallel.

3. If any three planes do not pass through the same line, then their three lines of intersection meet at a point, or are parallel.

4. Two intersecting straight lines cannot both be parallel to a third straight line.

THEOREM II

If a straight line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane in which they lie.

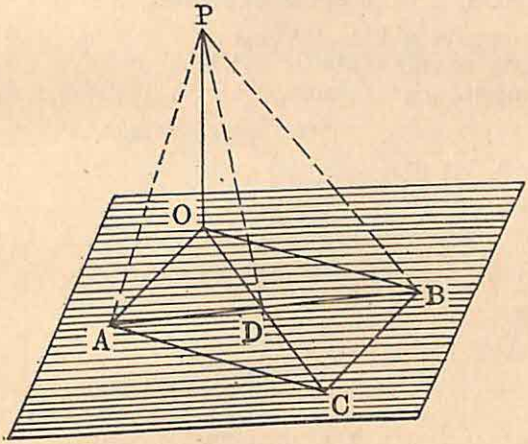


Fig. 9

Let OP be perpendicular to each of two intersecting lines OA, OB at their point of intersection at O.

It is required to prove that PO is perpendicular to the plane AOB.

Through O draw *any straight line* OC in the plane AOB. Also in the same plane, through C, draw CA parallel to OB, and CB parallel to OA, so that $OACB$ is a parallelogram. Join AB to meet OC in D, so that D is the mid-point of AB.

[∵ *diagonals of a parallelogram bisect each other*]

Join PA, PB, PD.

Proof. In the triangle PAB, the base AB is bisected at D.

$$\therefore \quad PA^2 + PB^2 = 2AD^2 + 2PD^2 \quad \cdots \quad (1)$$

Similarly from the triangle OAB, we get

$$OA^2 + OB^2 = 2AD^2 + 2OD^2 \quad \cdots \quad (2)$$

$\therefore$ subtracting (2) from (1),

$$(PA^2 - OA^2) + (PB^2 - OB^2) = 2(PD^2 - OD^2) \quad (3)$$

$$\left.\begin{array}{l}\text{Now, } PA^2 - OA^2 = OP^2\,; \\ \text{and } \;\; PB^2 - OB^2 = OP^2\end{array}\right\} \quad \cdots \quad \cdots \quad (4)$$

since, $\angle POA$, $\angle POB$ are right angles.

$\therefore$ from (3) and (4), we get

$$2OP^2 = 2(PD^2 - OD^2).$$

$\therefore \quad OP^2 + OD^2 = PD^2.$

$\therefore \quad \angle POD$ *i.e.*, $\angle POC$ is a right angle.

Thus, PO is perpendicular to *any line* OD which meets it in the plane AOB.

$\therefore \quad PO$ is perpendicular to the plane AOB. Q. E. D.

Alternative method of Proof.

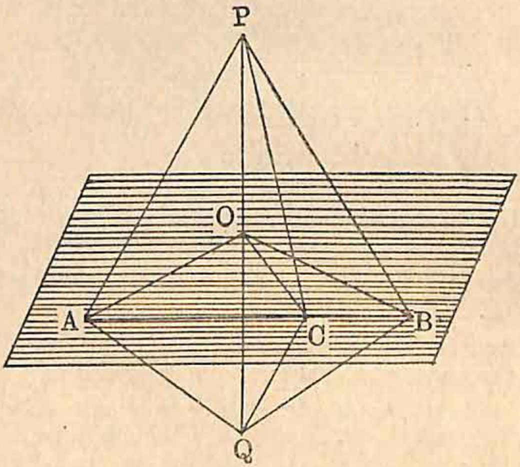


Fig. 10

Let PO be perpendicular to both OA, OB.

It is required to prove that PO is perpendicular to the plane of OA, OB *i.e.*, the plane AOB.

Join AB and in the plane of AOB draw *any straight line* OC, meeting AB in C. Then it will be sufficient to prove that PO is perpendicular to OC. Produce PO beyond the plane OAB to Q, making $OQ = OP$.

Join PA, PB, PC ; also QA, QB, QC.

Now AO being perpendicular bisector of PQ, $PA = QA$ and BO being perpendicular bisector of PQ, $PB = QB$.

Hence, $\triangle^s PAB$, QAB are congruent (AB being common to both). $\therefore\ \angle PAB = \angle QAB$.

Then in $\triangle^s$ PAC, QAC, $PA = QA$, AC is common and $\angle PAC = \angle QAC$. $\therefore\ \triangle^s PAC$, QAC are congruent. $\therefore\ CP = CA$.

Now in the $\triangle^s$, OPC, OQC, $OP = OQ$, $PC = QC$ and OC is common. $\therefore\ \triangle^s$ OPC, OQC are congruent. $\therefore\ \angle POC = \angle QOC$ which being adjacent angles on the straight line POQ, $\angle POC = \angle QOC =$ a right angle.

$\therefore\ PO$ is perpendicular to *any line* OC which meets it in the plane AOB.

$\therefore\ PO$ is perpendicular to the plane of OA, OB.

Q. E. D.

Exercises 2

1. Show that from a point in space, three straight lines can be drawn so that each is perpendicular to the plane of the other two.

2. A straight line is drawn through the centre of a circle perpendicular to the two radii OA, OB of the circle. Show that all points on the circumference of the circle are equidistant from any point on the line.

3. If O be a point in the plane of the triangle ABC and if P be a point outside the plane such that PO is perpendicular to OA and OB and if $PA = PB = PC$, show that O is the circum-centre of the triangle ABC.

4. If O be the circum-centre of any given triangle ABC and if P be any point outside the plane of the triangle ABC such that $PA=PB=PC$, show that PO is perpendicular to the plane ABC.

5. P is any point outside a given plane, and O, A, B, C, D are points in the plane such that $\angle POA = \angle POB =$ a right angle. If $PA=PB=PC=PD$, prove that the points A, B, C, D are concyclic. Find the centre of the circle, passing through A, B, C, D.

6. ABC is a triangle right-angled at C. P is a point outside the plane ABC, such that $PA=PB=PC$. If D be the mid-point of AB, prove that PD is perpendicular to CD and hence deduce that PD is perpendicular to the plane of the triangle.

THEOREM III

All straight lines drawn perpendicular to a given straight line at a given point on it are co-planar.

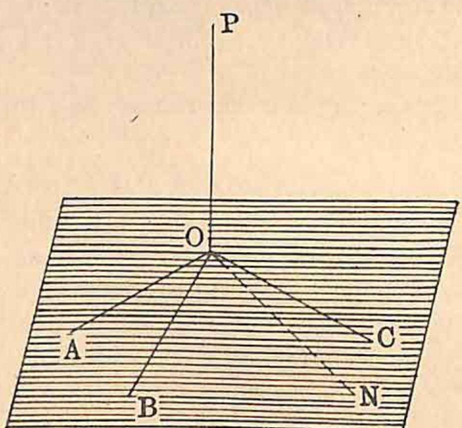


Fig. 11

Let PO be the given straight line and let the straight lines OA, OB, OC be drawn perpendicular to PO at O.

It is required to prove that OA, OB, OC are co-planar *i.e.*, they all lie in one plane.

Proof. Evidently OA, OB lie in one plane and PO, OC in another plane.

If possible, let the plane POC cut the plane AOB in the straight line ON.

Now, since PO is perpendicular to OA and OB, PO is perpendicular to the plane AOB.

But ON being in the plane AOB and meeting PO at O, PO is perpendicular to ON; hence $\angle PON$ is a right angle. Also $\angle POC$ is a right angle.

Now, PO, OC, ON lying in the same plane, $\angle POC$, $\angle PON$ cannot be both right angles. Hence, ON coincides with OC. Thus, OC is the line of intersection of the planes AOB and POC. Hence QC lies in the plane of OA, OB.

$\therefore$ OA, OB, OC are co-planar.

Similarly it can be easily shown that if other lines OD, OE, OF etc. are drawn perpendicular to PO at O, they lie in the plane of AOB. Hence, OA, OB, OC, OD, OE, OF etc. are all co-planar. Q. E. D.

Exercises 3

1. How many horizontal straight lines can be drawn through a given point of a vertical line and how do they lie ?

How many vertical lines can be drawn through a given point ?

2. Through the mid-point O (*i.e.*, the intersection of its diagonals) of a horizontal square $ABCD$, a vertical line OP is drawn. Show that PA, PB, PC, PD are all equal.

3. Prove that there cannot be more than three mutually perpendicular straight lines meeting at a point.

4. If a triangle revolves about its base, show that the vertex describes a circle.

5. Find the locus of a point in space—

(i) equidistant from two given points

(ii) equidistant from three given non-collinear points.

6. Find a point in a given straight line in space which is equidistant from two given points outside the line. When is this impossible ?

7. Prove that a point can be found in a plane equidistant from three points outside the plane. State the exceptional case, if any.

8. Show that there is one and only one point equidistant from four non-co-planar points, no three of which are collinear.

THEOREM IV

If of two parallel straight lines one is perpendicular to a plane, the other is also perpendicular to the same plane.

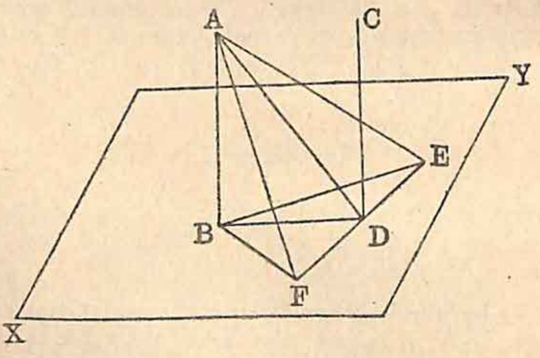


Fig. 12

Let AB, CD be two parallel straight lines of which AB is perpendicular to the plane XY.

It is required to prove that CD is also perpendicular to the plane XY.

Let B, D be the points at which the given lines meet the plane. Join AD, BD.

In the plane XY, draw the straight line EDF perpendicular to BD, making $DF = DE$.

Join AE, AF; BE, BF.

Proof. Since BD bisects EF at right angles.

$\therefore \quad BE = BF.$

Now in the $\triangle^s$ ABE, ABF, AB is common and $BE = BF$; also $\angle ABE = \angle ABF$, since each is a right angle because AB is perpendicular to the plane XY and BE, BF lie in the plane.

$\therefore \quad \triangle^s$ ABE, ABF are congruent.

Hence, $\quad AE = AF.$

Again in the $\triangle^s$ ADE, ADF, AD is common, $DE = DF$ and $AE = AF$.

$\therefore \quad \triangle^s$ ADE, ADF are congruent.

$\therefore \quad \angle ADE = \angle ADF$ and these being adjacent angles on the straight line FDE, FD is perpendicular to AD.

Since FD is perpendicular to DA and DB, FD is perpendicular to the plane DAB.

As AD, DB both lie in the plane of the parallel lines AB, CD, four lines AB, CD, AD, BD are co-planar. Hence, CD lies in the plane of ABD and thus FD is perpendicular to CD.

Since AB, CD are parallel and BD meets them,

$\therefore \quad \angle CDB + \angle ABD = 2$ right angles. But $\angle ABD =$ a right angle.

$\therefore \quad \angle CDB$ is a right angle.

Hence, CD is perpendicular to BD.

Thus, CD being perpendicular to both DB, DF, is perpendicular to the plane XY in which they lie.

$\therefore \quad CD$ is perpendicular to the plane XY. Q. E. D.

Cor. 1. It can be easily proved that the *converse of the theorem* is true; *i.e., if two straight lines are both perpendicular to a plane, then they are parallel.*

Let AB, CD be both perpendicular to the plane XY. Then with the same construction, it can be proved, as before,

that FD is perpendicular to DA. But FD is perpendicular to DB (*construction*) and is also perpendicular to DC, since FD lies in the plane XY and CD is perpendicular to the plane XY (*Hyp.*) Hence, DC lies in the plane of DB, DA [*Theo. III*]. But AB also lies in the same plane and hence AB, CD are co-planar and since $\angle ABD = \angle CDB$, each being a right angle, $\angle ABD + \angle CDB = 2$ right angles. Hence, AB is parallel to CD.

Cor. 2. Theorem of three perpendiculars.

If AB is perpendicular to a plane XY and if from B, the foot of the perpendicular, a line BC is drawn perpendicular to any straight line DE in the plane, then AC is also perpendicular to DE.

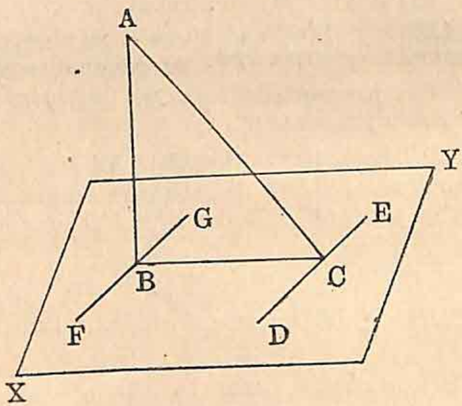


Fig. 13

Proof. Through B draw FG *in the plane* XY, parallel to DE. Since BC is perp. to DE, it is also perp. to FG. Again, AB being perp. to the plane XY, AB is perp. to FG. Thus FG, being perp. to AB, BC, is perp. to the plane ABC and hence DE being parallel to FG is also perp. to the plane ABC and hence to the line CA.

Thus, AC is perpendicular to DE.

Exercises 4

1. If the straight lines AB, CD are perpendicular in a plane meeting it at B, D and if AB, CD are equal in length and on the same side of the plane, show that $ABCD$ is a rectangle.

2. *Straight lines in space, which are parallel to a given straight line are parallel to one another.*

3. If the middle points of the adjacent sides of a skew quadrilateral are joined, prove that the figure so formed lies in one plane and form a parallelogram.

4. P is a point outside the plane of two parallel straight lines AB, CD. From the point P, PL is drawn perpendicular to AB and LM is drawn perpendicular to CD. Prove that PM is perpendicular to CD.

5. If AB, CD, EF are three equal, parallel straight lines not lying in one plane, and if their extremities form two triangles ACE, BDF, show that the triangles are congruent.

6. If perpendiculars are drawn from a point to a system of parallel straight lines in space, show that they lie on a plane perpendicular to the parallel lines.

THEOREM V

If a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to that plane.

Let the straight line AB be perpendicular to the plane XY and let LM be any plane passing through AB. [Fig. 14]

It is required to prove that the plane LM is perpendicular to the plane XY.

Draw BC in the plane XY perpendicular to LN, the line of intersection of the two planes XY and LM.

Now AB being perpendicular to the plane XY, it is perpendicular to BC and BN, as they lie in the plane XY. Hence, $\angle ABC$ is a righ-angle.

Also $\angle ABC$ measures the dihedral angle between the two planes XY and LM, since AB and BC are both perpendicular to the line of section LN.

$\therefore$ the plane LM is perpendicular to the plane XY.

Q.E.D.

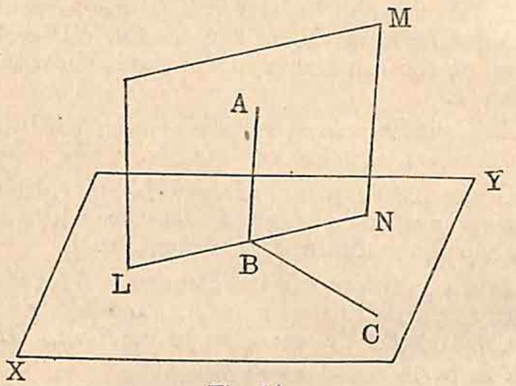


Fig. 14

Cor. If from any point in the line of intersection of two perpendicular planes, a line is drawn on either of the planes perpendicular to the line of intersection, then it is perpendicular to the other plane.

Exercises 5

1. Draw a plane perpendicular to a given plane and passing through a given straight line not lying in that plane.

2. Through a given point draw a plane perpendicular to each of two intersecting planes.

3. If two intersecting planes are each perpendicular to a third plane, their line of intersection is also perpendicular to that plane.

4. Show that the three straight lines in which three mutually perpendicular planes cut one another are themselves mutually perpendicular.

5. From a point A two straight lines AB, AC are drawn perpendicular one to each of two intersecting planes. Prove that the line of intersection of these two planes is perpendicular to the plane ABC.

6. Prove that planes perpendicular to a given plane cut one another in parallel straight lines.

CHAPTER III

VOLUMES AND SURFACE AREAS OF REGULAR SOLIDS

3·1. When any portion of space is bounded by one or more surfaces, it is called a *solid figure* or simply a *solid*. These surfaces are called the faces of the solid and the intersections of the adjacent faces are called its *edges*.

When a solid is bounded by plane faces, it is called a *polyhedron*. A polyhedron is said to be *regular*, if its faces are all regular, such as equilateral triangles, squares, etc.

3·2. Parallelopiped.

If a solid is bounded by three pairs of parallel planes, it is called a *parallelopiped*.

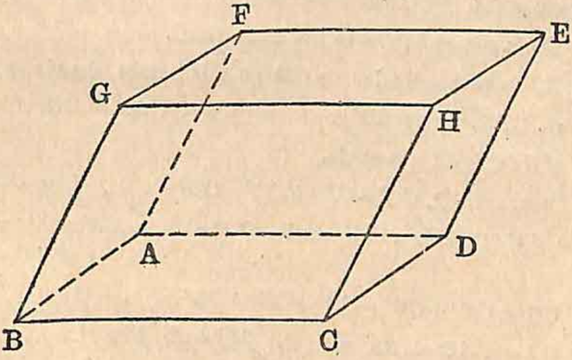


Fig. 15

Here $ABCDEFGH$ is a parallelopiped. Its faces are all parallelograms.

Rectangular parallelopiped.

If a parallelopiped has its faces all rectangles, it is called a *rectangular parallelopiped* (or a cuboid).

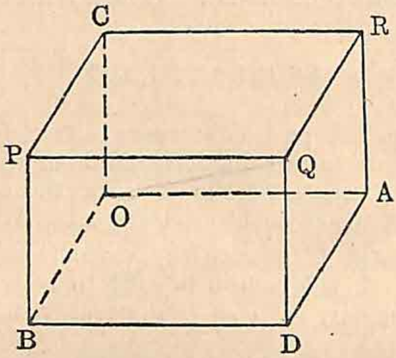


Fig. 16

Let $OADBPQRC$ be a rectangular parallelopiped. Here OA, OB, OC are three mutually perpendicular lines.

Here, $\angle COA$, $\angle COB$ are right angles.

$\therefore$ OC is perpendicular to the face $AOBD$.

$\therefore$ OC is perpendicular to OD, as OD lies in the plane $AOBD$.

Hence, QD being parallel to OC is perpendicular to OD.

$\therefore$ $\angle ODQ$ is a right angle,

$\therefore$ from $\triangle OQD$, $OQ^2 = OD^2 + DQ^2 = OD^2 + OC^2$, since $OC = DQ$.

Now $\angle OAD$ being a right angle, $OD^2 = OA^2 + AD^2 = OA^2 + OB^2$ (as $AD = OB$)

$\therefore$ $OQ^2 = OD^2 + OC^2 = OA^2 + OB^2 + OC^2$.

Let $OA = a$, $OB = b$, $OC = c$. $\therefore$ $OQ^2 = a^2 + b^2 + c^2$.

There are *three* pairs of rectangular faces parallel two by two, *viz.* ($PBDQ$, $COAR$), ($DARQ$, $BOCP$), ($CPQR$, $OBDA$). The opposite faces are congruent. The four diagonals are OQ, AP, BR, CD, and they are all equal.

Whole surface of the rectangular parallelopiped

$$= 2(bc + ca + ab).$$

Volume $= abc$

i.e. $=$ *length* $\times$ *breadth* $\times$ *height*

or $=$ (*area of the base*) $\times$ *height.*

3·3. Cube.

If all the sides of a rectangular parallelopiped are equal (*i.e.* if the bounding faces are all squares) then the parallelopiped is called a *cube.*

If a denote each side or edge of a cube, then

the whole surface area of the cube $= 6a^2$

and the volume $= a^3$ *i.e.*, $= (\text{edge})^3$.

3·4. Prism.

A solid bounded by plane faces of which the side-faces are parallelograms, and the two end-faces called the *ends* are two congruent parallel plane polygons, is called a *prism.*

The straight lines in which the side-faces intersect two by two are called the *side-edges* of the prism. The side-faces being all parallelograms, the side-edges are all parallel and equal and the number of side-faces is equal to the number of the sides of the polygon at the end of the prism. In the two ends of a prism are polygons, it is called a polygonal prism ; *e.g.*, if the two ends are triangles it is called a *triangular prism.*

Right Prism.

A solid bounded by plane faces of which the side-faces are rectangles, and the two end-faces are two congruent parallel plane polygons, is called a *right prism.*

For a right prism, the side-edges are perpendicular to its ends and its height is equal to its edge. The height of

a right prism is sometimes called its length. The adjoining figure is that of a right prism.

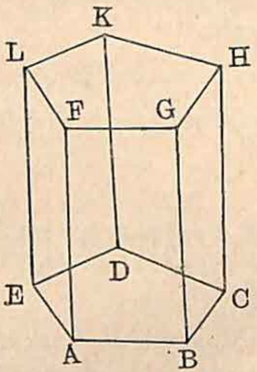


Fig. 17

(1) *Area of the lateral surface of a right prism*

= (perimeter of the base) × height.

(2) *Volume of a right prism*

= (area of the base) × height.

3·5. Pyramid.

A solid bounded by plane faces of which one, called the *base*, is any plane polygon, and the remaining faces are all triangles meeting in a point, is called a *pyramid*, the common point of the triangular faces being called its *vertex*. Obviously if the base polygon of a pyramid has n sides, the pyramid has n triangular faces. A pyramid is called a *square pyramid* or *triangular pyramid* according as the base is a *square* or a triangle. The *height* of a pyramid is the perpendicular distance from the vertex to the base. The straight lines in which the triangular faces intersect two by two are called its *edges* (or *slant edges*).

Right Pyramid.

When a solid is bounded by plane faces of which one called the *base* is a *regular polygon*, and the remaining faces

are all isosceles triangles meeting in a point (called the *vertex*), which lies on the straight line drawn perpendicular to the base from its centre (*i.e.*, the centre of the inscribed or circumscribed circle of the polygon), it is called a *right pyramid*.

For a right pyramid the edges of the base are all equal and the side-faces are equal isosceles triangles. The *slant height* of a right pyramid is the length of the perpendicular drawn from the vertex to any side of the base and hence bisecting it. Slant height is the same for each slant face. The *slant surface* of a right pyramid is the sum of its triangular faces. The *height* of a right pyramid is the length of the line joining the vertex to the centre of the base.

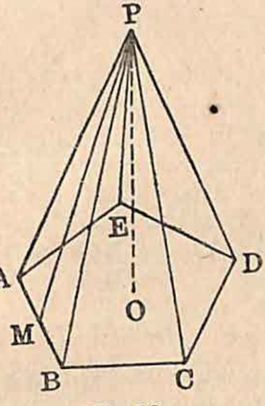


Fig. 18

The adjoining figure is that of a right pyramid.

The base is $ABCDE$, a regular pentagon here, of which O is the centre. P is the vertex, PO is the height. It has five triangular faces which are all equal. PM is the slant height, bisecting AB at right angles. The above pyramid is very often written as $(P, ABCDE)$.

(i) *Slant surface of a right pyramid*
$= \frac{1}{2}$**(perimeter of the base) × slant height.**

(ii) *Volume of a right pyramid*
$= \frac{1}{3}$**(area of the base) × height.**

Note. The *whole* surface = slant surface + area of the base.

3·6. Tetrahedron.

A solid bounded by four triangular faces is called a *tetrahedron*. One of the triangular faces being taken as the base, the point where the other three meet is called the *vertex* of the tetrahedron. If all the four faces of a tetrahedron are equal equilateral triangles, the tetrahedron is said to be *regular*. All the edges of a regular tetrahedron are equal. A tetrahedron is thus a *triangular pyramid*.

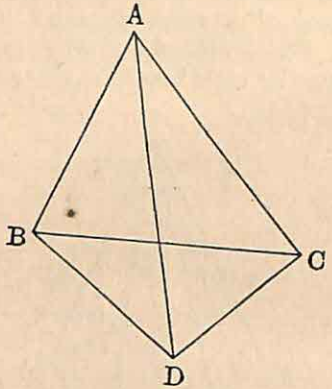


Fig. 19

The adjoining figure is a tetrahedron. It has four triangular faces, ABC, ACD, ADB, BCD and six sides or edges AB, BD, DC, CA, AD, BC. The pairs (AB, CD), (BD, AC), (AD, BC) are called *opposite edges*. A is the vertex and BCD is the *base*. The length of the perpendicular from the vertex A upon the face BCD is called its *height*.

(i) *Whole surface of the tetrahedron*

= sum of the areas of the four faces.

(ii) *Volume of the tetrahedron*

$=\frac{1}{3}$(area of the base) × height.

3·7. Right circular cone.

If a solid is generated by the complete revolution of a right-angled triangle about one of the sides containing

the right angle as axis, the solid is called a *right circular cone.*

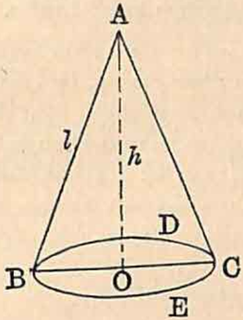


Fig. 20

The adjoining figure is that of a right circular cone. It is generated by the complete revolution of the right-angled triangle AOB, about the side AO as axis. The hypotenuse AB generates the curved surface of the cone and is called the generating line or the generator of the surface. AB is also called *slant height* of the cone and is usually denoted by l. The other side OB describes the circle $BDCE$ of radius OB and centre O, which is called the base of the cone. The *radius of the base*, OB, is usually denoted by r. The point A is called the *vertex* of the cone and the length of its axis AO is called its *height* and is usually denoted by h. The angle $\angle BAC$ is called the *vertical angle* of the cone, and $\angle OAC$ or $\angle OAB$ is called the semi-vertical angle of the cone.

If h be the height, r the radius of the base, l the slant height and α the semi-vertical angle of the cone,

(i) *Area of the curved surface (or lateral surface) of a right circular cone,*

$$= \tfrac{1}{2}(\text{circumference of the base}) \times \text{slant height}$$
$$= \tfrac{1}{2} \times 2\pi r \times l = \pi r l$$
$$= \pi r\sqrt{h^2 + r^2} = \pi r^2 \operatorname{cosec} \alpha.$$

(ii) *whole surface* $= \pi r\,(l + r)$.

(iii) *volume* $= \frac{1}{3}$ (area of the base) $\times$ height

$$= \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi h^3 \tan^2\alpha.$$

Note. We easily have from the right-angled triangle AOB

$$r = h \tan\alpha\ ;\ l = r \operatorname{cosec}\alpha,\ l = \sqrt{h^2 + r^2}.$$

Also if v_1, v_2 be the volumes of two right circular cones of heights h_1, h_2 and if they have the same vertical angle 2α,

then, $v_1 : v_2 = \frac{1}{3}\pi h_1^3 \tan^2\alpha : \frac{1}{3}\pi h_2^3 \tan^2\alpha = h_1^3 : h_2^3$.

Thus, *the volumes of cones with the same vertical angle are to one another as the cubes of their heights.*

3·8. Right circular cylinder.

The solid generated by the complete revolution of a rectangle about one of its sides as axis is called a *right circular cylinder.*

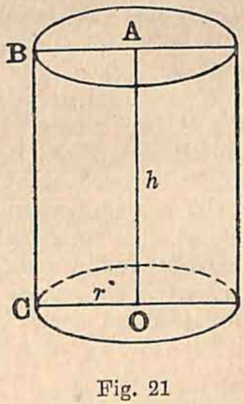


Fig. 21

The adjoining figure is that of a right circular cylinder. It is generated by the revolution of the rectangle $OABC$ about the side AO. AO is called the *axis* of the cylinder. The opposite side BC generates the curved surface of the cylinder and is called the *generating line* of the cylinder. The side AB describes a *circular end* with A as centre and AB as radius and the opposite side OC describes

a *circular end* with O as centre and OC as radius. Both the circular ends are called *bases* and they are equal in area. The length of the axis OA is called the *height* of the cylinder. OA is also sometimes called the length of the cylinder. If r be the radius of the base and h the height of a right circular cylinder, then

(i) *area of the curved surface*

$= \textbf{(circumference of the base)} \times \textbf{height}$

$= \mathbf{2\pi rh.}$

(ii) *area of the whole surface*

$=$ area of the curved surface $+$ area of the two ends

$= 2\pi rh + 2\pi r^2$

$= \mathbf{2\pi r(h+r).}$

(iii) *volume* $= \textbf{(area of the base} \times \textbf{height)}$

$= \mathbf{\pi r^2 h.}$

3·9. Sphere.

If s solid is generated by the revolution of a semi-circle about its diameter as axis, it is called a *sphere*.

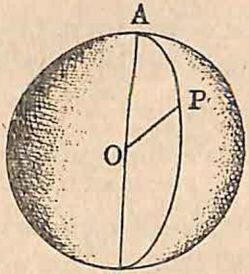


Fig. 22

The adjoining figure is that of a sphere generated by the complete revolution of the semi-circle APB about the diameter AB as axis and the semi-circumference APB

describes the surface of the sphere. O is the centre of the semi-circle and OP its radius.

As the semi-circumference revolves, all the points in it remain at a constant distance from the centre O.

Thus, *a sphere may be defined as the locus of a point which moves in space in such a way that its distance from a fixed point remains constant.*

The fixed point O is called the *centre* of the sphere and the constant distance OP is called its *radius*.

If r be the radius of the sphere,

$$\textit{area of the surface of the sphere} = 4\pi r^2$$

$$\textit{volume of the sphere} = \tfrac{4}{3}\pi r^3.$$

Note. Half of a sphere is called a hemisphere.

3·10. Illustrative Examples.

Ex. 1. *If the area of the four walls of a room is* 1024 *sq. ft. and the height is* 16 *feet, find the perimeter of the floor.*

Let a, b, c be the length, breadth and height of the room. Then $c = 16$ ft.

Then, $2c\,(a+b) = 1024$ sq. ft.,

i.e., $2(a+b) = \dfrac{1024}{c} = \dfrac{1024}{16}$ ft. $= 64$.

$\therefore$ the perimeter of the floor $= 64$ ft.

Ex. 2. *Find the area of the lateral surface of a right prism whose ends are squares of sides of length* 3 *inches and whose height is* 1 *ft.* [C. *U.*]

Area of the lateral surface = (perimeter of the base) × height

$= 1$ ft. $\times 1$ ft. $= 1$ sq. ft.

Ex. 3. *Find the volume of the right pyramid in which the base is a triangle whose sides are* 8 *cm*, 15 *cm.*, 17 *cm.. and the height is* 12 *cm.* [*C. U.*]

Since, $8^2 + 15^2 = 17^2$, the triangle is a right-angled triangle, of which the hypotenuse is of length 17 c.m.

$\therefore$ area of the triangle $= \frac{1}{2} \times 8 \times 15 = 60$ sq. cm.

Hence, the volume of the pyramid $= \frac{1}{3}$(area of the base) × height

$= \frac{1}{3} \times 60 \times 12 = 240$ cu. cm.

Ex. 4. *Find the whole surface of a regular tetrahedron, the length of each edge of which is $2a$.*

Let $ABCD$ be the tetrahedron of which A is the vertex and BCD is the base. Since BCD is an equilateral triangle of side $2a$, its area $= \sqrt{3a \times a \times a \times a}$ [by using the formula for the area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$] $= \sqrt{3}a^2$.

$\therefore$ whole surface = sum of the areas of the four faces, which are here all equal, being equal to the area of the base $BCD = 4 \times \sqrt{3}.a^2$ $= 4a^2\sqrt{3}$.

Ex. 5. *Find the area of the curved surface and the volume of a right circular cone which is 15 ft. high and the radius of whose base is 8 ft.* [*C. U.*]

Let l be the slant height of the cone; then $l = \sqrt{15^2 + 8^2} = 17$ ft. Area of the curved surface of the cone $= \pi r l$

$= 3{\cdot}1416 \times 8 \times 17 = 427{\cdot}3$ sq. ft. (approx.).

Volume of the cone $= \frac{1}{3}\pi r^2 h$, where h is the height of the cone

$= \frac{1}{3} \times 3{\cdot}14 \times 8^2 \times 15$

$= 1005{\cdot}31$ cu. ft.

Ex. 6. *The curved surface of a right circular cylinder is 2000 sq. cm. and the diameter of the base is 40 cm.; find the volume and the height of the cylinder.*

If r be the radius of the base and h the height of the right circular cylinder,

area of the curved surface $= 2\pi r h = 2000$, and $2r = 40$.

$\therefore\ r = 20$. $\quad \therefore\ \pi h = 50$ cm.

Now, volume $V = \pi h r^2 = 50 \times 400 = 20{,}000$ cu. cm.

Ex. 7. *How many solid spheres, each 6 cm. in diameter, could be moulded from a solid metal right circular cylinder whose height is 45 cm. and diameter 4 cm.?* [*C. U.*]

Let r_1 be the radius of the sphere; then its volume $= \frac{4}{3}\pi r_1^3$.

Here $r_1 = 3$ cm. $\therefore$ volume of the sphere $= \frac{4}{3}\pi . 3^3$.

Let r_2 be the radius and h the height of the cylinder; then its volume $= \pi r_2^2 h = \pi \times 2^2 \times 45$ (since $r_2 = 2$, $h = 45$).

Let n be the required number of solid spheres moulded.

$\therefore\ n \times \frac{4}{3}\pi \times 3^3 = \pi \times 2^2 \times 45$.

$\therefore\ n = 5$.

Thus, 5 spheres can be moulded.

Examples on Chapter III

Sec. A : (*On rectangular parallelopipeds*)

1. The whole surface of a rectangular block is 52 sq. ft. The base contains 6 sq. ft., and one vertical face contains 12 sq. ft. Find the height of the block.

2. The perimeter of the floor of a room is 24 cm., and the total area of the four walls is 144 sq. cm. Find the height of the room.

3. The length, breadth and height of a rectangular block are in the ratio 5 : 6 : 7 ; and the whole surface of the block is 1926 sq. cm. Find the height of the block.

4. The diagonal of a rectangular parallelopiped is 13 cm. The area and perimeter of the floor are 12 sq. cm.and 14 cm. respectively. Find the volume of the parallelopiped.

5. The length, breadth and height of a closed box are 10 cm., 9 cm., 7 cm. respectively, and the total inner surface is 262 sq. cm. If the walls of the box are uniformly thick, find the thickness.

Sec. B : (*On right Prisms*)

1. The area of the lateral surface of a right prism is 80 sq. in. If the base of the prism be a square of side 4 in., find the height of the prism.

2. Show that the volume of a right prism of height h standing on an equilateral triangle of side a is $\frac{\sqrt{3}}{4}a^2h$.

3. The base of a right prism is a trapezium whose parallel sides are 7 cm. and 3 cm., the distance between them being 4 cm. If the volume of the prism be 200 cu. cm., find the height of the prism.

4. Through a wooden pipe, whose cross-section is a square on a side of 4 cm., water flows uniformly at the rate of 50 cm. per sec. How long will it take to discharge 48 litres ?

5. Two prisms of equal height are such that the magnitude of the area of the base of one prism is double the perimeter of the base of the other prism. Show that the magnitude of the volume of the first prism is double the magnitude of the area of the lateral surfaces of the other prism.

Sec. C : (*On right Pyramids*)

1. A right pyramid of height $1''$ stands on a square base of side $4''$. Find the slant height and the slant edge.

2. Find the volume of a right pyramid 12 cm. high which stands on a rectangular base of sides 10 cm. and 8 cm.

3. A right pyramid stands on a rectangular base whose sides are $6'$ and $8'$, and the length of each slant edge is $13'$, Find the height of the pyramid.

4. Find the volume of the right pyramid whose base is a triangle of which the sides are 13 ft., 14 ft., 15 ft. and whose height is 20 ft.

5. A right pyramid stands on a square base each of whose sides is 12 ft. and the slant faces are equilateral triangles. Find the height and the volume of the pyramid.

6. OA, OB, OC are three mutually perpendicular lines in space, and $OA=a$, $OB=b$, $OC=c$. Prove that the volume of the pyramid $=\frac{1}{6}abc$.

7. OA, OB, OC are three mutually perpendicular lines of equal length a. Find the area of the triangle ABC.

Sec. D : (*On right circular cones*)

1. Find the volume and the area of the slanting surface of a right circular cone of height 4 feet and the radius of whose base is 3 feet ($\pi=\frac{22}{7}$).

2. If S be the area of the curved surface and α the semi-vertical angle, h the height and r the radius of the base of a right circular cone, prove that

$$S=\frac{\pi h^2 \sin \alpha}{\cos^2 \alpha}=\frac{\pi r^2}{\sin \alpha}.$$

3. What is the length of the canvas 3 feet wide which will be required to make a conical tent which is 28 ft. high and which covers an area of 154 sq. yds. ? ($\pi = \frac{22}{7}$)

4. Show how to draw a plane parallel to the base of a right circular cone, so that it divides the cone into two parts of equal curved surfaces. [*C. U.*]

5. Show how to draw a plane parallel to the base of a right circular cone, so that it divides the cone into two parts of equal volumes.

6. A right circular cone 20 ft. high has its upper part cut off by a plane passing through the middle point of its axis. If the plane of section be at right angles to the axis, and if the radius of the base of the original cone be 4 feet, find the volume of the truncated cone. [*C. U.*]

7. If S_1, S_2 be the curved surfaces and h_1, h_2 the heights of two right circular cones with the same vertical angle, show that

$$S_1 : S_2 = h_1^2 : h_2^2.$$

8. The upper portion of a right circular cone cut off by a plane parellel to the base is removed. If the curved surface of the remainder be $\frac{3}{4}$ of that of the whole cone, show that the cutting plane bisects the altitude of the cone.

Sec. E : (*On right circular cylinders*)

1. If the volume of a right circular cylinder be 1980 cu. ft. and the area of its curved surface be 660 sq. ft., find the radius of the base and the height of the cylinder.

2. If the height of a right circular cylinder be 15·8 ft. and radius of the base be 4·2 ft., find the whole surface of the cylinder. ($\pi = 3·1416$)

3. Find the height and the volume of the cylinder, the curved surface of which is 2000 sq. ft. and the diameter of whose base is 20 ft.

$$[\text{ Given } \frac{1}{\pi} = 0·31831]$$

4. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of the base is to the height as 3 : 4.

5. A right prism stands on a square base whose side is 7·2 cm. Find the volume of a right circular cylinder whose height is 3 cm. and whose base touches the four sides of the square, the centre of the base being at the centre of the square.

Sec. F : (*On spheres*)

1. Three solid golden balls of radii 3, 4 and 5 millimeters are melted into one single solid golden ball. Find the radius of the single ball. [*C. U.*]

2. A lump of clay in the form of a solid sphere is converted into a right circular cylinder of height 16 inches. Find the radius of the base of the cylinder supposing it to be equal to the radius of the sphere. [*C. U.*]

3. A sphere and a right circular cylinder of the same radius have equal volumes. By what percentage does the diameter of the cylinder exceed its heigbt ? [*C. U.*]

4. The weights of two balls are in the ratio of 5 to 11 and the weights of a cubic foot of the material in the two balls are in the ratio of 121 to 25. Find the ratio of their radii.

5. If a solid sphere of radius 4 ft. is blown into a hollow sphere, the radius of whose external surface is 5 ft., show that the thickness of the hollow sphere, assuming it to be uniform, is approximately 1·06 ft.

[Given $\sqrt[3]{(61)} = 3{\cdot}94$]

6. The volumes of a sphere and of a right circular cylinder are as 4 : 9, and the radius of the base of the cylinder is equal to three times the radius of the sphere. Show that the radius of the sphere is three times the height of the cylinder.

7. A sphere of diameter 6 cm. is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cm. If the sphere be completely submerged, by how much will the surface of the water be raised ?

8. A right circular cylinder is circumscribed about a hemisphere and a right circular cone is inscribed in the cylinder, such that its vertex is at the centre of one end of the cylinder and its base coincides with the other end of the cylinder, show that

$$\frac{\text{Vol. of cone}}{1} = \frac{\text{Vol. of hemisphere}}{2} = \frac{\text{Vol. of cylinder}}{3}.$$

ANSWERS

Sec. A :—(1) 4 ft. (2) 6 cm. (3) 21 cm. (4) 144 cu. cm. (5) 1 cm.

Sec. B :—(1) 5 inches. (3) 10 cm. (4) 1 minute.

Sec. C :—(1) $2''{\cdot}236$, $3''$. (2) 960 cu. cm. (3) 12 ft. (4) 560 cu. ft.

(5) 8·48 ft. ; 407·04 cu. ft. (7) $\frac{\sqrt{3}}{2} \cdot a^2$.

Sec. D :—(1) $37\frac{5}{7}$ cu. ft. ; $47\frac{1}{7}$ sq. ft. (3) 770 ft.

(4) the plane divides the height in the ratio $\sqrt{2}-1 : 1$.

(5) The plane divides the height in the ratio $\sqrt[3]{2}-1$.

(6) $293\frac{1}{3}$ cu. ft.

Sec. E :—(1) radius = 6 ft. ; height = 17·5 ft. (2) 527·79 sq. ft.

(2) 31·8 ft. (5) 122·2 cu. cm.

Sec. F :—(1) 6 m. m. (2) 12 inches. (3) 50%. (4) 5 : 11.

(7) 1 cm.

CALCUTTA UNIVERSITY

PRE-UNIVERSITY EXAMINATION PAPERS

1961

1. (*a*) The co-ordinates of A, B, C are $(-1, 5)$, $(3, 1)$ and $(5, 7)$ respectively. D, E, F are the mid-points of BC, CA, AB respectively. Calculate the area of the triangle DEF.

(*b*) Obtain the equation of the straight line through the point $(2, 1)$ and perpendicular to the line joining the points $(2, 3)$ and $(3, -1)$.

2. (*a*) Obtain the equation of the locus of a point which moves in the plane of (x, y) in such a way that its distance from the point $(2, 3)$ is always two-thirds of its distance from the y-axis.

(*b*) Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x', y').

3. (*a*) Show that the centres of the following three circles are in a straight line:

$$x^2 + y^2 - 2x - 6y - 5 = 0, \qquad x^2 + y^2 - 4x - 10y - 7 = 0,$$
$$x^2 + y^2 - 6x - 14y - 9 = 0.$$

(*b*) Find the eccentricity and the co-ordinates of the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

4. (*a*) Obtain the equations of the lines which bisect the angle between the lines

(i) $a_1x + b_1y + c_1 = 0$ and (ii) $a_2x + b_2y + c_2 = 0$.

(*b*) Obtain the equation of the circle which has its centre at the point $(3, 4)$, and touches the straight line $5x + 12y = 1$.

5. If a straight line is perpendicular to each of two intersecting straight lines at their point of intersection, shew that it is also perpendicular to the plane in which they lie.

6. (*a*) Find the locus of a point in space equidistant from two given points.

(*b*) Three solid spheres of gold whose radii are 1 cm., 6 cms. and 8 cms. respectively are melted into a single gold sphere. Find the radius of the sphere so formed.

1962

1. (*a*) Find the angle between the two straight lines $y=mx+c$; $y=m'x+c'$.

(*b*) Obtain the equations to the straight lines each of which passes through the point $(2, -1)$ and intersects the axes of co-ordinates at points equidistant from the origin and calculate the angle between them.

2. (*a*) Obtain the equation to the circle which passes through the points $(2, -1)$ and $(3, -2)$ and has its centre on the straight line $2x+4y-3=0$.

(*b*) A conic is represented by the equation $4x^2-9y^2=36$; calculate its eccentricity, length of latus rectum and the co-ordinates of the foci.

3. (*a*) Shew that the chord of the parabola $y^2=4ax$, whose equation is $y-x\sqrt{2}+4a\sqrt{2}=0$, is a normal to the parabola, and find the co-ordinates of the point of the parabola at which it is the normal.

(*b*) Find the possible values of k for which the straight line $3x+4y=k$ may touch the circle $x^2+y^2=10x$.

4. (*a*) Find the equation to the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point x', y'.

(*b*) A point P moves in the plane of (x, y) in such a way that its distance from the lines $12x+5y-4=0$ and $3x+4y+7=0$ are equal; obtain the equation to the locus traced out by P.

5. (*a*) Shew that all straight lines drawn perpendicular to a given straight line at a given point on it are coplanar.

(*b*) The diagonal of a rectangular block is 10 cms., and the sum of the lengths of its edges is 80 cms.; calculate the total area of the outer surface of the block.

6. (*a*) Shew that if a straight line is perpendicular to a plane, then every plane passing through it is also perpendicular to the plane.

(*b*) A right circular cone is 10 cms. high and its slant height is 15 cms. Calculate the volume of the cone. [$\pi=22/7$]